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On the smallest disks enclosing graph-directed fractals

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ABSTRACT

We consider the graph-directed iterated function systems and give upper bounds for the diameters of the smallest disks enclosing their attractors. We also give an algorithm to obtain these smallest enclosing disks with any proximity.

1. Introduction

Iterated function system theory is one of the main branches of fractal geometry which has numerous applications such as image compression [3], signal modeling and wavelets [18,24], neural networks [22], chaotic systems [1,2]. An iterated function system (abbreviated as IFS) on a complete metric space X is a finite collection of contractive mappings $\{\varphi_i : X \to X\}_{i=1}^N$ with contractivity factors $0 \le \lambda_i < 1$. By the Hutchinson theorem [13], one can state that there exists a fixed point of the operator $\Phi : \mathcal{H}(X) \to \mathcal{H}(X)$, $\Phi(U) = \bigcup_{i=1}^N \varphi_i(U)$, where $\mathcal{H}(X)$ denotes the complete metric space of nonempty compact subsets of X. This fixed point $\mathcal{A}$ is called the attractor of the IFS. Moreover, $\lim_{n\to\infty} \Phi^n(U) = \mathcal{A}$ for any compact set $U \in \mathcal{H}(X)$.

Determining the smallest disk that encloses the attractor is needed in various fields such as approximation of the attractors (many algorithms require a certain information of a disk bounding the attractor) [12], numerical multifractal analysis which can be used for understanding the structure of complex networks [9,14,17,21], and more accurate computation of box-counting dimension the calculation techniques of which are still in development [2,6,10,15].

In the previous works, there are several algorithms to obtain the smallest enclosing disk. In 1991, Hart and DeFanti [11] proposed an algorithm to produce a disk that covers the attractor of an IFS $\{\varphi_i\}_{i=1}^N$. Starting from an arbitrary disk D_0, a sequence of disks (D_n) is determined iteratively, where the center of each D_n is $c_n = \frac{1}{K}\sum_{i=1}^K \varphi_i(c_{n-1})$ with radius $r_n = \max_{i=1,\dots,K}\{|\varphi_i(c_{n-1}) - c_n| + \mathrm{diam}(\varphi_i(D_{n-1}))/2\}$. Although it is not guaranteed that the disk D_n includes the attractor, the limit disk contains it. Dubuc and Hamzaoui [7] defined a radius function depending on a point in $\mathbb{R}^d$, which will be the center, so that the disk covers the attractor. Also, they proved the existence of the global minimum of this function and gave an upper bound for it. A different approach can be found in [4]: instead of covering the attractor with only one disk, Canright used a collection of disks D_i with centers x_i (the fixed points of maps) and the radii satisfying $r_i = \lambda_i \max_{i\neq j}(|x_i - x_j| + r_j)$. The union of the disks in the collection, called the envelope, contains the attractor, i.e. $\mathcal{A} \subset \cup_{i=1}^N D_i$. Edalat et al. [8] introduced an algorithm with an approach similar to [7]. To find the optimal center that minimizes the radius of the disk, they used the Langrange multipliers method. In [20], Rice improved Hart and DeFanti's approach to obtain the disk with minimal radius by implementing various algorithms such as Downhill Simplex method. Another related work with the problem of enclosing the attractor is [5], Chu and Chen used a bounding box to enclose $\mathcal{A}$ instead of using bounding disk.

In [16], Martyn developed a novel algorithm to find the smallest enclosing disk of an IFS attractor. The methods mentioned above have a lack of accuracy in some cases (e.g. fern) but Martyn's approach allows one to approximate the smallest disk at any precision. In the paper, an initial axis-aligned box (AAB) that bounds the attractor is given. Starting from this initial AAB, he creates a family

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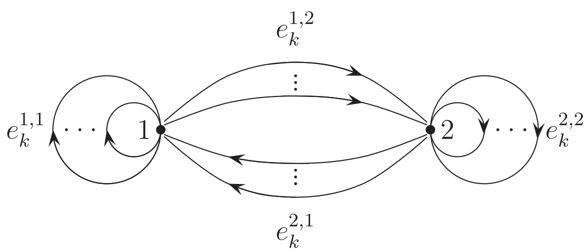

Fig. 1. A graph with two nodes.

of AABs with each diameter not greater than $\varepsilon > 0$ and their union containing $\mathcal{A}$. Once such a family is obtained, then the so-called the spanning point set $\Gamma(\varepsilon, \delta)$ is formed for ε and suitably chosen δ, so that the smallest disk containing these points also contains the attractor $\mathcal{A}$ with proximity depending on ε and δ. Finally, the smallest disk enclosing the finite number of spanning points is found by Welzl's algorithm. The details of Martyn's method can be found in Section 3.

Welzl's algorithm [23] is a recursive algorithm that aims to compute the smallest enclosing disk of finite number of points in d-dimensional space. For a set $P = \{p_1, p_2, \dots, p_n\}$ of n-points in $\mathbb{R}^d$, let md(P) denote the smallest closed disk containing all points in P. The algorithm starts with $P' = \varnothing$ then adds the points from P individually to the set P' and computes $md(P')$. After the disk $D = md(P')$ is computed for $P' = \{p_1, p_2, \cdots, p_i\}$ with $1 \le i \le n$, the algorithm checks whether $p_{i+1} \in D$. If this is the case, it controls the next point; otherwise p_{i+1} must be on the boundary of the smallest disk of the set $\{p_1, p_2, \dots, p_{i+1}\}$. Then, another subroutine starts to compute $md(\{p_1, p_2, \dots, p_{i+1}\})$ with p_{i+1} on its boundary. The algorithm stops once all points of P are considered.

Fractals can be considered in a different aspect with the graph self-similarity. It can be identified as a finite collection of metric spaces, each of them being finite union of contractive copies of themselves. This type of spaces are called graph-directed fractals. In literature, the smallest enclosing disk problem is not considered in the graph-directed case. This paper aims to address this gap.

We now briefly summarize the graph-directed IFS. Let $\{(X_\alpha, d_\alpha) \,|\, \alpha = 1, \dots, N\}$ be a finite collection of complete metric spaces. The so called *graph-directed fractals* $\mathcal{A}^\alpha \subset X_\alpha$ can be defined as

$$\mathcal{A}^\alpha = \bigcup_{\beta=1}^{N} \bigcup_{k=1}^{K^{\alpha,\beta}} \varphi_k^{\alpha,\beta}(\mathcal{A}^\beta)$$

where $\varphi_k^{\alpha,\beta} : X_\beta \to X_\alpha$ are contractive mappings with contractivities $0 \le \lambda_k^{\alpha,\beta} < 1$ $(\alpha, \beta = 1, \dots, N$ and $k = 1, 2, \dots, K^{\alpha,\beta})$. The system $\{X_\alpha, \varphi_k^{\alpha,\beta}\}$ is called a graph-directed iterated function system (GIFS).

The mapping relationship of $\{X_\alpha, \varphi_k^{\alpha,\beta}\}$ can be coded by a graph $G = (V, E)$ where $V = \{1, 2, \dots, N\}$ is a vertex (node) set and E is an edge set. Each vertex α corresponds to a space (X_α, d_α) and each edge $e^{\alpha,\beta} \in E$ between vertices α and β corresponds to a contraction between the spaces X_α and X_β (but in the reverse direction), see Fig. 1. This is why we use the term "graph-directed iterated function system".

One can define an operator Φ on the product of the spaces $\mathcal{H}(X_\alpha)$ in the following way:

$$\begin{aligned}\Phi : \mathcal{H}(X_1) \times \mathcal{H}(X_2) \times \cdots \times \mathcal{H}(X_N) &\to \mathcal{H}(X_1) \times \mathcal{H}(X_2) \times \cdots \times \mathcal{H}(X_N)\\ (U_1, \dots, U_N) &\mapsto (\Phi_1(U_1, \dots, U_N), \dots, \Phi_N(U_1, \dots, U_N))\\ &= \left(\bigcup_{\beta=1}^{N} \bigcup_{k=1}^{K^{1,\beta}} \varphi_k^{1,\beta}(U_\beta), \dots, \bigcup_{\beta=1}^{N} \bigcup_{k=1}^{K^{N,\beta}} \varphi_k^{N,\beta}(U_\beta) \right).\end{aligned}$$

This map is also a contraction on $\mathcal{H}(X_1) \times \mathcal{H}(X_2) \times \cdots \times \mathcal{H}(X_N)$ and thus there exists subsets $\mathcal{A}^\alpha$ of X_α, called attractors of the system $\{X_\alpha, \varphi_k^{\alpha,\beta}\}$ for $\alpha = 1, \dots, N$. Moreover, $\lim_{n\to\infty} \Phi^n(U_1, U_2, \dots, U_N) = (\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^N)$ for any compact set $U_\alpha \in \mathcal{H}(X_\alpha)$ (see [9,18,19]).

This paper consists of two independent parts. In the first part, we will give upper bounds for the diameters of the smallest disks enclosing the attractors of a GIFS and in the second part, we will generalize the so-called "spanning point algorithm" to GIFS and show how to find disks as close as desired to the smallest disks.

2. Upper bounds for the radii of smallest disks containing the attractors

In this section, we will give upper bounds for the radii of the smallest disks enclosing the attractors of a GIFS. Our argument is based on the following lemma:

Lemma 1. *Let* $\{X_\alpha, \varphi_k^{\alpha,\beta}\}$ $\left(\alpha, \beta = 1, 2, \dots, N; k = 1, 2, \dots, K^{\alpha,\beta}\right)$ *be a GIFS. If there exist subsets* $U_\alpha \subseteq X_\alpha$ *such that* $\Phi_\alpha(U_1, U_2, \dots, U_N) \subseteq U_\alpha$ *then* $\mathcal{A}^\alpha \subseteq U_\alpha$ *where* $\mathcal{A}^\alpha$ *are the attractors of the system.*

Proof. On employing the hypothesis of the lemma, we write

$$\begin{aligned}\Phi(U_1, \dots, U_N) &= (\Phi_1(U_1, \dots, U_N), \dots, \Phi_N(U_1, \dots, U_N))\\ &\subset (U_1, \dots, U_N)\end{aligned}$$

(here, note that the inclusion is componentwise) and applying the operator Φ successively we obtain $\Phi^n(U_1, U_2, \dots, U_N) \subset (U_1, U_2, \dots, U_N)$ for all $n \in \mathbb{N}$. Then, we get $\mathcal{A}^\alpha \subset U_\alpha$, since $\lim\limits_{n\to\infty} \Phi^n(U_1, U_2, \dots, U_N) = (\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^N)$. □

One can use Lemma 1 to acquire disks centered at an arbitrary $x_\alpha \in X_\alpha$ with radius r_{x_α} such that $\mathcal{A}^\alpha \subset B(x_\alpha, r_{x_\alpha})$. The following proposition determines those radii:

Proposition 2. *Let* $\{X_\alpha, \varphi_k^{\alpha,\beta}\}$ $\left(\alpha, \beta = 1, 2, \dots, N; k = 1, 2, \dots, K^{\alpha,\beta}\right)$ *be a GIFS with attractors* $\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^N$*. Let* $(x_1, x_2, \dots, x_N) \in X_1 \times X_2 \times \cdots \times X_N$ *be given. If the radii* $r_{x_1}, r_{x_2}, \dots, r_{x_N}$ *satisfy a number of* $K := \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} K^{\alpha,\beta}$ *inequalities*

$$r_{x_\alpha} - \lambda_k^{\alpha,\beta} r_{x_\beta} \ge d_\alpha(\varphi_k^{\alpha,\beta}(x_\beta), x_\alpha), \quad k = 1, 2, \dots, K \tag{1}$$

then we have $\mathcal{A}^\alpha \subseteq B(x_\alpha, r_{x_\alpha})$*.*

Proof. For $\alpha = 1, 2, \dots, N$, if we can show that

$$\Phi_\alpha(B(x_1, r_{x_1}), \cdots, B(x_N, r_{x_N})) \subseteq B(x_\alpha, r_{x_\alpha})$$

then we are done by Lemma 1. To prove this inclusion, we need to show that

$$\varphi_k^{\alpha,\beta}(B(x_\beta, r_{x_\beta})) \subseteq B(x_\alpha, r_{x_\alpha})$$

for $\beta = 1, 2, \dots, N$ and $k = 1, 2, \dots, K^{\alpha,\beta}$. Let $x_0 \in B(x_\beta, r_{x_\beta})$. Using (1),

$$\begin{aligned}d_\alpha(\varphi_k^{\alpha,\beta}(x_0), x_\alpha) &\le d_\alpha(\varphi_k^{\alpha,\beta}(x_0), \varphi_k^{\alpha,\beta}(x_\beta)) + d_\alpha(\varphi_k^{\alpha,\beta}(x_\beta), x_\alpha)\\ &\le \lambda_k^{\alpha,\beta} d_\beta(x_0, x_\beta) + d_\alpha(\varphi_k^{\alpha,\beta}(x_\beta), x_\alpha)\\ &\le \lambda_k^{\alpha,\beta} r_{x_\beta} + d_\alpha(\varphi_k^{\alpha,\beta}(x_\beta), x_\alpha) \le r_{x_\alpha}.\end{aligned}$$

Then $\varphi_k^{\alpha,\beta}(x_0) \in B(x_\alpha, r_{x_\alpha})$ which completes the proof. □