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On the number of limit cycles bifurcated from some Hamiltonian systems with a non-elementary heteroclinic loop

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ABSTRACT

In this paper, we study the bifurcation of limit cycles in two special near-Hamiltonian polynomial planer systems which their corresponding Hamiltonian systems have a heteroclinic loop connecting a hyperbolic saddle and a cusp of order two. In these systems, we will compute the asymptotic expansions of corresponding first order Melnikov functions near the loop and the center to analyze the number of limit cycles. Moreover, in the first system, by using the Chebychev criterion, we study the Poincaré bifurcation.

1. Introduction

Let $H=H(x,y)$ be a polynomial function in $(x,\ y)$ of degree $m>1$. Consider the Hamiltonian system

$$\dot{x}=H_y(x,y),\quad \dot{y}=-H_x(x,y), \tag{1}$$

which has a continuous family of periodic orbits L_h parameterized by the values $h\in(a,\ b)$ and defined by the equation $H(x,y)=h$. Now take a small perturbation of (1) of the form

$$\dot{x}=H_y(x,y)+\varepsilon p(x,y,\delta),\quad \dot{y}=-H_x(x,y)+\varepsilon q(x,y,\delta), \tag{2}$$

where p and q are real polynomials of $(x,\ y)$ with $\max\{\deg p,\deg q\}=n>1$, which their coefficients are denoted by δ. Here, $\delta\in D\subset\mathbb{R}^k$ which D is a compact set and ε is a small positive parameter. Arnold in [1] asks about the maximum number of isolated zeros of the Abelian Integrals

$$M(h,\delta)=\oint_{L_h} qdx-pdy, \tag{3}$$

which is also known as the first order Melnikov function associated to system (2). This problem in the general case is called the week Hilbert's 16th problem which is closely related to the second part of Hilbert's 16th problem. Indeed, as a consequence of Poincaré–Pontryagin theorem, if $M(h,\ \delta)$ is not identically zero, then the total number of the limit cycles of (2) bifurcating from the annulus $\bigcup_{h\in(a,b)}\{L_h\}$ is bounded by the maximum number of isolated zeros of $M(h,\ \delta)$ for $h\in(a,\ b)$. One of the interesting situations for the family $\{L_h\}$ is when its inner boundary is a singular point of center type and its outer boundary is a non-elementary graphic. So the limiting behavior of $M(h,\ \delta)$ as $h\to a$ or $h\to b$ becomes important and there are many papers on this subject. In other words, there have been many studies on the limit cycle bifurcations near the boundaries of $\{L_h\}$. For example, Han et al. [5,7] studied $M(h,\ \delta)$ near a homoclinic loop with a saddle or a cusp of order 1. Atabaigi et al. [2], by using the method developed in [7], studied $M(h,\ \delta)$ near a homoclinic loop with a cusp of order 2. In the heteroclinic case with two saddles, a saddle and a cusp

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of order 1, or two cusps of order 1, the number of limit cycles was studied by Han et al. [5] and Sun et al. [10]. Li et al. [8] studied the number of limit cycles of the system (2) near a heteroclinic loop with two cusps of order 1 or 2.

Remark 1.1. Suppose that the unperturbed Hamiltonian system (1) has a homoclinic loop passing through a nilpotent singular point at the origin. Thus, the function $H(x, y)$ satisfies $H_x(0,0) = H_y(0,0) = 0$, and

$$\frac{\partial(H_y, -H_x)}{\partial(x,y)} \neq 0, \quad \det\frac{\partial(H_y, -H_x)}{\partial(x,y)} = 0.$$

Then, without loss of generality, we may suppose $H_{yy}(0,0) = 1$, $H_{xy}(0,0) = H_{xx}(0,0) = 0$. It follows that the expansion of $H(x, y)$ at the origin has the form

$$H(x,y) = \frac{1}{2}y^2 + \sum_{i+j\geq 3} h_{ij}x^iy^j. \tag{4}$$

By the implicit function theorem, there exists a change of variables that puts (4), again using the original variables, into the following form (see [7])

$$H(x,y) = H_0^*(x) + y^2\tilde{H}(x,y), \qquad H_0^*(x) = \sum_{j\geq 3} h_jx^j, \quad \tilde{H}(0,0) = \frac{1}{2}.$$

The coefficients h_j for $j = 3, \dots, 6$ are as follows

$$\begin{aligned} h_3 &= h_{30}, \quad h_4 = -\frac{1}{2}h_{21}^2 + h_{40}, \quad h_5 = h_{12}h_{21}^2 - h_{21}h_{31} + h_{50}, \\ h_6 &= -2h_{12}^2h_{21}^2 - h_{03}h_{21}^3 + h_{21}^2h_{22} \\ &\quad + 2h_{12}h_{21}h_{31} - \frac{1}{2}h_{31}^2 - h_{21}h_{41} + h_{60}. \end{aligned}$$

Let $k \geq 3$ be an integer such that

$$h_k \neq 0, \quad h_j = 0, \quad \text{for } j < k. \tag{5}$$

We have the following definition from Han et al. [7]:

Definition 1.2. (see [7]) Suppose that the unperturbed system (1) has a homoclinic loop given by $H(x,y) = 0$ passing through the origin. It is called a cuspidal homoclinic loop of order m provided (4) and (5) hold with $k = 2m+1$, $m \geq 1$.

Bakhshalizadeh et al. in [3] studied the Melnikov function $M(h, \delta)$ near the heteroclinic loop connecting a cusp of order two and a saddle point, and gave the asymptotic expansion of the first-order Melnikov function near this heteroclinic loop for (2) and then discussed the number of limit cycles of some polynomial Liénard systems. More precisely, they considered the following assumptions:

(A) The system (1) has a heteroclinic loop denoted by

$$L_0 := \{(x,y) : H(x,y) = 0\} = L_1 \bigcup L_2 \bigcup \{S_1, S_2\},$$

where L_1 and L_2 are heteroclinic orbits connecting singular points S_1 and S_2 so that $\omega(L_1) = \alpha(L_2) = S_2$ and $\omega(L_2) = \alpha(L_1) = S_1$.

(B) In a neighborhood of L_0 there is a family of periodic orbits of (1) denoted by $L_h = \{(x,y) : H(x,y) = h\}$ for $0 < -h \ll 1$.

Then for the expansion of $M(h, \delta)$ near $h = 0$, they proved the following theorem.

Theorem 1.3 [3]. *Consider the analytic system (2) and suppose (1) satisfies the assumptions (A) and (B). Then near $h = 0$ corresponding to the heteroclinic loop L_0, the Melnikov function $M(h, \delta)$ of system (2) has the following asymptotic expansion:*

$$\begin{aligned} M(h,\delta) &= c_0(\delta) + B_{00}c_1(\delta)|h|^{\frac{7}{10}} + B_{10}c_2(\delta)|h|^{\frac{9}{10}} + c_3(\delta)h\ln|h| \\ &\quad + c_4(\delta)\,h + B_{50}c_5(\delta)|h|^{\frac{11}{10}} + B_{30}c_6(\delta)|h|^{\frac{13}{10}} \\ &\quad - \frac{1}{17}B_{00}c_7(\delta)|h|^{\frac{17}{10}} - \frac{1}{19}B_{10}c_8(\delta)|h|^{\frac{19}{10}} \\ &\quad + c_9(\delta)h^2\ln|h| + O(h^2), \end{aligned} \tag{6}$$

in which

$$\begin{aligned} &c_0(\delta) = M(0,\delta) = \oint_{L_0} qdx - pdy|_{\varepsilon=0} = \sum_{i=1}^{2}\int_{L_i}(qdx - pdy)|_{\varepsilon=0}, \\ &c_1(\delta) = c_1(S_1,\delta), \quad c_2(\delta) = c_2(S_1,\delta), \quad c_3(\delta) = c_1(S_2,\delta), \\ &c_5(\delta) = c_4(S_1,\delta), \quad c_6(\delta) = c_5(S_1,\delta), \quad c_7(\delta) = c_6(S_1,\delta), \\ &c_8(\delta) = c_7(S_1,\delta), \quad c_9(\delta) = c_3(S_2,\delta), \end{aligned} \tag{7}$$

where $c_i(S_1, \delta)$, $i = 1, 2, 4, 5, 6, 7$ are given in Lemma 3.3 of [2] and $c_i(S_2, \delta)$, $i = 1, 3$ come from Lemma 3.1 in [5]. Finally,

$$\begin{aligned} c_4(\delta) &= \sum_{k=1}^{2}\int_{L_{0k}}(p_x + q_y - \sigma_k)|_{\varepsilon=0}dt \\ &\quad + \int_{L_{03}}(p_x + q_y)|_{\varepsilon=0}\,dt + b_1c_1(\delta) + b_2c_2(\delta) + b_3c_3(\delta), \end{aligned} \tag{8}$$

provided $b_{11} + 2a_{20}|_{S_2} = 0$ where $\sigma_k = (p_x + q_y)|_{S_k}$, $L_{0k} = L_0 \bigcap U_k$, $L_{03} = L_0 - (L_{01} \bigcup L_{02})$. In particular, if $c_1 = c_2 = c_3 = 0$, then

$$c_4(\delta) = \oint_{L_0}(p_x + q_y)|_{\varepsilon=0}\,dt = \sum_{k=1}^{2}\int_{L_k}(p_x + q_y)|_{\varepsilon=0}\,dt. \tag{9}$$

Moreover, by using Theorem 1.3 they studied the number of limit cycles near the heteroclinic loop and near the center.

Theorem 1.4 [3]. *Suppose that system (1) satisfies the conditions of Theorem 1.3. If there exists some $\delta_0 \in \mathbb{R}^k$, such that*

$$\begin{aligned} &c_0(\delta_0) = c_1(\delta_0) = \cdots = c_{k_1-1}(\delta_0) = 0, \quad c_{k1}(\delta_0) \neq 0, \\ &b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{k_2-1}(\delta_0) = 0, \quad b_{k_2}(\delta_0) \neq 0 \end{aligned}$$

and

$$\text{rank}\frac{\partial(c_0, c_1, \dots, c_{k_1-1}, b_0, b_1, \dots, b_{k_2-1})}{\partial(\delta_1, \cdots, \delta_k)} = k_1 + k_2,$$

then (2) can have $k_1 + k_2 + \frac{1-\text{sgn}(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycles for some (ε, δ) near $(0, \delta_0)$ from which k_1 limit cycles are near the heteroclinic loop, k_2 limit cycles are near the center and

$$\frac{1 - \text{sgn}(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$$

limit cycle is located between them, where $h_1 = 0 - \varepsilon_1$, $h_2 = 0 + \varepsilon_2$ with ε_1 and ε_2 are positive and very small.

Our aim in this paper is to investigate the bifurcation of limit cycles from two special Hamiltonian systems in the plane which are not Newtonian and have a heteroclinic loop connecting a hyperbolic saddle and a cusp of order two. The first system is a non-Newtonian system as follows:

$$\begin{aligned} \dot{x} &= \frac{38416}{177241}y(x^2+1)^2, \\ \dot{y} &= \frac{7}{6}x^6 + \frac{9}{2}x^5 + \frac{165}{28}x^4 + \frac{225}{98}x^3 - \frac{10125}{10976}x^2 \\ &\quad - \frac{50625}{76832}x - \frac{76832}{177241}y^2x^3 - \frac{76832}{177241}y^2x. \end{aligned} \tag{10}$$

The phase portrait of system (10) is shown in Fig. 1. In Section 2 we perturb this system by $\varepsilon y f(x)\frac{\partial}{\partial y}$, where $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5$ and then we study