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Response of non-linear oscillator driven by fractional derivative term under Gaussian white noise

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ABSTRACT

This paper aimed to investigate the stationary response of non-linear system with fractional derivative damping term under Gaussian white noise excitation. The corresponding Fokker–Plank–Kolmogorov (FPK) equation can be deduced by utilizing the stochastic averaging method and Stratonovich–Khasminskii theorem in the first place. And then we can solve the FPK equation to obtain the stationary probability densities (SPDs) of amplitude, which in fact can be used to describe the response of system. Furthermore, the analytical results coincide with the Monte Carlo results. Finally, one found that reducing fractional derivative order is able to enhance the response of system and increasing fractional coefficient can weaken the response of system. So the fractional derivative damping term has a great effect on the response of Duffing-Van der Pol oscillator. In addition, the response can also be influenced by other system parameters.

1. Introduction

The fractional-order derivative was first proposed by L'Hospital in 1695. And since then, fractional calculus has been widely investigated and applied in many fields as a kind of non-linear factors [1–5]. In contrast to the conventional integer-order systems, fractional-order derivative systems have some significant advantages: Firstly, in the past few years, a large amount of models in engineering vibrations indicated that long-memory factor exists in many actual systems, which are difficult to be precisely described by integer-order model [6]. So the fractional calculus was introduced to make up for this shortcoming. Bagley and Torvik [7] proved that half-order fractional derivative models can quite well simulate the frequency dependence damping of viscoelastic materials. In addition, an increasing number of facts [8–12] showed that the model with fractional derivative could give a more accurate description and give a deeper insight into the inherent nature of realistic physical systems than the integer-order model. Studying fractional dynamic has great theoretical significance and applicable value.

To obtain the approximate analytical solution of the fractional-order systems, stochastic averaging method has been extensively adopted due to the following two advantages: First, the system equations can be much simplified while the essential behaviors of system were retained. Second, the dimension of equations was often reduced, so that we can obtain the solution of the equation more easily. For instance, response of the fractional Duffing-Rayleigh system under Gaussian white noise was considered by Zhang et al. [13] employing stochastic averaging procedure. Yang et al. [14] investigated the stochastic response of the Van der Pol oscillator with two kinds of fractional derivative under Gaussian white noise factor utilizing stochastic averaging method. Yang [15,16] also studied the stationary response and stochastic response of non-linear systems and a class of self-excited systems with Caputo-type fractional derivative driven by Gaussian white noise. Chen and Zhu [17] discussed the stochastic jump and bifurcation of the Duffing oscillator endowed with fractional derivative damping of order α $(0<\alpha<1)$ combined harmonic and white noise excitations. Using the stochastic averaging method, Huang and Jin [18] demonstrated the stochastic response and stability of a single-degree-of-freedom non-linear system endowed with fractional derivative. Chen et al. [19] explored the first passage failure of multi-degree-of-freedom quasi-integrable Hamiltonian system with damping described by a fractional derivative.

There are multitudinous techniques were adopted to pick up the approximate solution analytically except for stochastic averaging measure. For example, the present authors [20,21] researched the response of the stochastic non-linear oscillators with fractional derivative damping term via Lindstead-Poincare (L-P) method and

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the multiple scales approach. Combining the advantages of (L-P) method and multiple scales approach, a new technique is proposed to deal with strongly non-linear stochastic systems with fractional derivative damping in Ref. [22]. Agrawal [23] proposed an analytical approach for stochastic dynamic systems with fractional derivative using eigenvector expansion method and Laplace transform. A frequency-domain way is applied to the stochastic systems having fractional derivative by Spanos and Zeldin in Ref. [24]. Cai and Lin [25] adopted the cumulant neglect closure scheme to evaluate the response spectral densities for nonlinear systems excited by Gaussian white noises or filtered Gaussian white noises.

Up to now, the effect of fractional derivative damping term on response of Duffing-Van der Pol system has not been studied. In this paper we try to explore the response of this system. The structure of this paper is organized as follows. In Section 2, the fractional Duffing-Van der Pol system under Gaussian white noise is introduced. We derive approximate SPDs via stochastic averaging method in Section 3. In Section 4, Monte Carlo simulation is used to verify the theoretical results and the effects of system parameters on response are discussed. Finally conclusion is listed in Section 5.

2. Duffing-Van der Pol system

In this section, the considered fractional-order Duffing-Van der Pol system subject to Gaussian white noise excitation can be given as

$$\ddot{X} + (\beta_1 + \beta_2 X^2)\dot{X} + \lambda D^\alpha X(t) + {\omega_0}^2 X + \gamma X^3 = W(t), \tag{1}$$

here, $\alpha (0 < \alpha < 1)$ indicates the order of fractional damping term; β_1 is the linear damping coefficient of the system; β_2 is the non-linear damping coefficient; ω_0 is the natural frequency of the system; λ denotes the coefficient of fractional derivative term; $g(x) = {\omega_0}^2 X + \gamma X^3$represents restoring force, which is a strongly non-linear function; $W(t)$ refers to Gaussian white noise with zero mean and correlation function $\langle W(t)W(t-\tau)\rangle = 2D\delta(\tau)$, here $\delta(\tau)$ is the Dirac-delta function .

Here, we adopt the Riemann–Liouville type fractional derivative

$$D^\alpha X(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{X(t-\upsilon)}{\upsilon^\alpha} d\upsilon. \tag{2}$$

3. Stochastic averaging method

According to the generalized harmonic functions the generalized displacement and velocity can be written as [26]

$$X(t) = A(t)\cos\Theta(t), \tag{3a}$$

$$\dot{X}(t) = -A(t)\eta(A,\Theta)\sin\Theta(t), \tag{3b}$$

in which

$$\Theta(t) = \Phi(t) + \Gamma(t), \tag{4}$$

$$\begin{aligned}\eta(A,\Theta) &= \frac{d\Phi}{dt}\\ &= \sqrt{\frac{2[U(A) - U(A\cos\Theta)]}{A^2\sin^2\Theta}}\\ &= \left[({\omega_0}^2 + 3\gamma A^2/4)(1 + \rho\cos 2\Theta)\right]^{1/2}\end{aligned} \tag{5}$$

where

$$\rho = \gamma A^2/(4{\omega_0}^2 + 3\gamma A^2), \tag{6}$$

$$U(X) = \frac{1}{2}{\omega_0}^2 X^2 + \frac{1}{4}\gamma X^4. \tag{7}$$

$\Theta(t)$ is the instantaneous phase and $\eta(A, \Theta)$ is the instantaneous frequency of oscillation. $U(X)$ denotes the potential energy; $\cos\Theta(t)$ and $\sin\Theta(t)$ are so called generalized harmonic functions.

Expanding $\eta(A, \Theta)$ into Fourier series, one can arrive at

$$\eta(A,\Theta) = b_0 + \sum_{i=1}^{\infty} b_{2i}(A)\cos(2i\Theta), \tag{8}$$

where

$$b_{2i}(A) = \frac{1}{2\pi}\int_0^{2\pi} \nu(A,\Theta)\cos(2i\Theta)d\Theta \quad i = 0, 1, 2, 3\cdots. \tag{9}$$

the approximate averaged frequency $\varphi(A)$ can be obtained in the following form

$$\varphi(A) = ({\omega_0}^2 + 3\gamma A^2/4)^{1/2}(1 - \rho^2/16) = b_0(A). \tag{10}$$

Treating Eqs.(3a) and (3b) as a generalized Van der Pol transformation from $(X, \dot{X})$ to (A, Γ),one can receive the following stochastic differential equations of amplitude A and phase Γ:

$$\frac{dA}{dt} = M_{11}(A,\Gamma) + M_{12}(A,\Gamma) + G_1(A,\Gamma)W(t), \tag{11a}$$

$$\frac{d\Gamma}{dt} = M_{21}(A,\Gamma) + M_{22}(A,\Gamma) + G_2(A,\Gamma)W(t), \tag{11b}$$

where

$$\begin{aligned}M_{11} &= \frac{A\eta(A,\Theta)\sin\Theta}{g(A)}\lambda D^\alpha(A\cos\Theta),\\ M_{12} &= \frac{(\beta_1 + \beta_2 A^2\cos^2\Theta)A^2\eta^2(A,\Theta)\sin^2\Theta}{g(A)},\\ M_{21} &= \frac{\eta(A,\Theta)\cos\Theta}{g(A)}\lambda D^\alpha(A\cos\Theta),\\ M_{22} &= -\frac{(\beta_1 + \beta_2 A^2\cos^2\Theta)A\eta^2(A,\Theta)\sin\Theta\cos\Theta}{g(A)},\\ G_1 &= -\frac{A\eta(A,\Theta)\sin\Theta}{g(A)},\\ G_2 &= -\frac{\eta(A,\Theta)\cos\Theta}{g(A)}.\end{aligned} \tag{12}$$

Eq. (11) can be modeled as Stratonovich stochastic differential equation and then transformed into *Itô* stochastic differential equation by adding Wong Zakai correction term [27]. The result is

$$dA = m(A)dt + \sigma(A)dB(t), \tag{13}$$

and the drift and diffusion coefficients, respectively, are

$$m(A) = \left\langle M_{11} + M_{12} + D\frac{\partial G_1}{\partial A}G_1 + D\frac{\partial G_1}{\partial \Gamma}G_2\right\rangle_\Theta, \tag{14}$$

$$\sigma^2(A) = \langle 2DG_1G_2\rangle_\Theta. \tag{15}$$

where $\langle\bullet\rangle_\Theta$ represents the averaging with respect to Θ from 0 to 2π.

Next we require calculating the above drift and diffusion coefficients. Firstly, one can obtain the approximate expression of the $\Theta(t)$ by substituting Eq.(10) into Eq.(4), as follows

$$\Theta(t) \approx \phi(A)t + \Gamma(t). \tag{16}$$

And then on account of A and Γ vary slowly with time, so the following approximate relation can be received via Eq. (16)

$$\Theta(t-\tau) \approx \Theta(t) - \phi(A)\tau. \tag{17}$$