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Review

Dynamics of a delayed predator-prey system with stage structure and cooperation for preys

Soumen Kundu*, Sarit Maitra

Department of Mathematics, National Institute of Technology, Durgapur 713209, India

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ABSTRACT

In this paper we have taken a delayed predator-prey system with stage structure among prey species. We assume that there is cooperation among the mature and immature preys to ensure immature prey's existence and there is a maturation delay for immature to be mature for the preys. With this model the local stability has been discussed in presence of delay. By taking maturation delay as the key parameter, the condition for local stability has been obtained by constructing a Lyapunov functional. By taking the maturation delay as the bifurcation parameter the necessary conditions for Hopf-bifurcation have been discussed both analytically and numerically. The length of delay has been estimated to preserve the stability. Also, Lyapunov exponents have been evaluated numerically for different cases.

1. Introduction

Among the models of mathematical ecology, predator-prey interactions are one of the most important ones. Lotka-Volterra first introduced a mathematical model on predator-prey system [14]. After that lot of researchers have studied predator-prey models by incorporating different situations, like: intra-species competition [6,9], cooperation [16] stage structure [7,18,25] etc. In classical population model it is assumed that all the populations have the same ability [24] and which is unrealistic in the considerations that the new born specimens are immediately able to compete and reproduce [10]. In order to include the effect it is assumed that in real life newly born first have to grow up, i.e., in a real world, almost all the populations have the life history that can be divided into two stages, immature and mature, where the immature populations are raised by their parents (mature populations) [20,24]. Aiello and Freedman [2] proposed and studied the stage-structured single species model with time delay as:

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t), \end{aligned} \tag{1}$$

where $x_i(t)$ is the density of the immature and $x_m(t)$ is the density of the mature populations at time t, respectively. α is the birth rate of the immature population, γ and β are the death rates of the immature and the mature populations respectively, τ is the maturity delay, $\alpha e^{-\gamma\tau} x_m(t-\tau)$ represents the quantity which the immature, born at time $t-\tau$ can survive at time t. Based on the above ideas, many authors studied different kinds of ecological models with stage structure for more than one species models [5,19,21–23]. Gourley and Kuang [8] have discussed the stage structured predator-prey model with constant maturation time delay and with this model the stability and the bifurcation have been discussed. Bandyopadhyay and Banerjee [3] have discussed a stage structured prey-predator model with stage structure for preys. Also, they have assumed the time delay for the predators due to gestation. Using the model the stability and conditions for Hopf-bifurcation have been discussed there.

In this paper we have discussed a predator-prey system with stage structure on preys. We assume that the birth rate of the immature preys depend in its feeding by the mature prey and it grows logistically [12,13]. The death rate of the immature and mature preys are proportional to the existing immature and mature preys respectively. Also, we assume that there is cooperation among mature and immature preys to protect immature preys against the predators. The predators consumes both the preys (immature and mature) as their food. The death rate of predators are proportional to the existing predators and there is intra-species competition among the predators. As an example for this model we assume a subsystem of a forest consisting of buffalos as preys and lions as predators. Based on the above assumptions we derive

* Corresponding author.
E-mail addresses: soumenkundu75@gmail.com (S. Kundu), sarit2010.nt@gmail.com (S. Maitra).

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the model as:

$$\frac{dx_1}{dT} = rx_2(1-\frac{x_1+x_2}{K}) - d_1x_1 - bx_1 + \sigma x_1x_2 - c_1x_1y_1,$$
$$\frac{dx_2}{dT} = bx_1 - d_2x_2 - c_2x_2y_1,$$
$$\frac{dy_1}{dT} = -d_3y_1 - Dy_1^2 + e_1x_1y_1 + e_2x_2y_1, \quad (2)$$

where x_1, x_2 and y_1 are the densities of immature preys, mature preys and predators respectively and $r>0$, is the birth rate of the immature preys, $K>0$ is the carrying capacity for the preys, d_1, d_2, $d_3>0$ are death rates of the immature, mature and the predators respectively. $b>0$ is the conversion rate from immature to mature preys, $\sigma>0$ is the cooperation rate among the immature and mature preys, c_1, $c_2>0$ are the rate of predation for immature and mature preys respectively. $D>0$ is the intra-species competition rate for predators, e_1, $e_2>0$ are the conversion rates from prey mass to predator mass. Now substituting $x=\frac{x_1}{K}$, $y=\frac{x_2}{K}$, $z=\frac{(c_2+c_1)y_1}{e_1+e_2}$, $T=\frac{t}{r}$ the non-dimensional form of system (2) is given as

$$\frac{dx}{dt} = y(1-y) - \alpha_1x - \beta x + \gamma_1xy - \delta_1xz,$$
$$\frac{dy}{dt} = \beta x - \alpha_2y - \delta_2yz,$$
$$\frac{dz}{dt} = -D_1z - D_2z^2 + \delta_3xz + \delta_4yz, \quad (3)$$

where $\alpha_1=\frac{d_1}{r}$, $\beta=\frac{b}{r}$, $\gamma_1=\frac{\sigma K}{r}-1$, $\delta_1=\frac{c_1(e_1+e_2)}{r(c_1+c_2)}$, $\alpha_2=\frac{d_2}{r}$, $\delta_2=\frac{c_2(e_1+e_2)}{r(c_1+c_2)}$, $D_1=\frac{d_3}{r}$, $D_2=\frac{D(e_1+e_2)}{r(c_1+c_2)}$, $\delta_3=\frac{e_1K}{r}$, $\delta_4=\frac{e_2K}{r}$.

Introducing time delay for maturation in the preys equations the system (3) becomes

$$\frac{dx}{dt} = y(1-y) - \alpha_1x - \beta_1(\tau)x(t-\tau) + \gamma_1xy - \delta_1xz,$$
$$\frac{dy}{dt} = \beta_1(\tau)x(t-\tau) - \alpha_2y - \delta_2yz,$$
$$\frac{dz}{dt} = -D_1z - D_2z^2 + \delta_3xz + \delta_4yz, \quad (4)$$

subject to the initial conditions

$$x(\theta) = \phi_1(\theta) > 0, \theta\in[-\tau,0); \phi_1(0) > 0,$$
$$y(\theta) = \phi_2(\theta) > 0,$$
$$z(\theta) = \phi_3(\theta) > 0, \quad (5)$$

where β_1 is a function of delay τ. For numerical analysis of our model we have taken a particular form of $\beta_1(\tau)$ from ref. [1].

In this paper, first we shall show that the solutions of (4) are positive and bounded. Then we shall derive the equilibrium points (Section 3) for the system (4). In Section 4, the stability analysis of (4) has been discussed in presence of delay by constructing a suitable Lyapunov functional and the conditions for local stability have been derived. In Section 5, the conditions for Hopf-bifurcation have been derived by taking the maturation delay (τ) as bifurcation parameter. In Section 6 we estimate the length of delay to preserve the stability and Section 7 deals with the numerical results and their interpretation. Finally, the paper ends with a conclusion of our work.

2. Positivity and boundedness of solutions

In this section we shall discuss about the positivity and boundedness of the solutions of (4). The system (4) can be written in the form

$$\dot{X} = F(X), \quad (6)$$

where $X=(x,y,z)^T\in R^3$ and $F(X)$ is given as

$$F(F_1(X),F_2(X),F_3(X))^T =$$
$$\begin{bmatrix} y(1-y)-\alpha_1x-\beta_1x(t-\tau)+\gamma_1xy-\delta_1xz \\ \beta_1x(t-\tau)-\alpha_2y-\delta_2yz \\ -D_1z-D_2z^2+\delta_3xz+\delta_4yz \end{bmatrix}.$$

Now let $R_+^3=[0,\infty)^3$ and $F: R_+^{3+1}\to R^3$ satisfies locally Lipschitz's condition and $[F_i(X)]_{x(t)=0,y(t)=0,z(t)=0,X\in R_+^3}\geq 0, i=1,2,3.$ Following the steps of [3] we can say that the solution of (6) as well as the solutions of (4) with the unique positive initial conditions are positive and each component of X remains in the interval [0, A) for some $A>0$. If $A=\infty$ then $\lim_{t\to\infty}\sup(x(t)+y(t)+z(t))=\infty$.

Now we prove that the solutions of (4) are bounded. Let us construct a function $V(x, y, z)$ as

$$V = x+y+z. \quad (7)$$

Differentiating (7) w.r.t. t we get,

$$\frac{dV}{dt} \leq y - \alpha_1x\left(1-\frac{\gamma_1}{\alpha_1}y\right) + xz(\delta_3-\delta_1) + yz(\delta_4-\delta_2). \quad (8)$$

If

$$\delta_3<\delta_1, \delta_4<\delta_2, y<\frac{\alpha_1}{\gamma_1}, \quad (9)$$

then from (8) we can write

$$\frac{dV}{dt} \leq y \leq (x+y+z). \quad (10)$$

This gives us the following result

$$(x+y+z) \leq (x(0)+y(0)+z(0))e^{-t}. \quad (11)$$

If $0<\phi_1(\theta)+\phi_2(\theta)+\phi_3(\theta)<M$, $\theta\in[-\tau,0]$ and M is any positive constant, then $0<x(0)+y(0)+z(0)<M$, hence we can say that the solutions of (4) are bounded if (9) holds. □

3. Equilibrium points

There are several equilibrium points of system (4) like: the trivial equilibrium $E_1(0, 0, 0)$, the predator free equilibrium $E_2(\bar{x},\bar{y},0)$ where $\bar{x},\bar{y}$ are the solutions of

$$\bar{y}(1-\bar{y}) - \alpha_1\bar{x} - \beta_1\bar{x} + \gamma_1\overline{xy} = 0,$$
$$\beta_1\bar{x} - \alpha_2\bar{y} = 0, \quad (12)$$

and the interior equilibrium point $E_3(x^*, y^*, z^*)$, where the form of x^*, y^* and z^* can be obtained by solving the following set of equations:

$$y^*(1-y^*) - \alpha_1x^* - \beta_1x^* + \gamma_1x^*y^* - \delta_1x^*z^* = 0,$$
$$\beta_1x^* - \alpha_2y^* - \delta_2y^*z^* = 0, \quad (13)$$
$$-D_1z^* - D_2z^{*2} + \delta_3x^*z^* + \delta_4y^*z^* = 0.$$

Throughout this work we shall discuss the dynamics of the interior equilibrium point E_3 unless otherwise it is specified.

4. Stability analysis

Linear stability implies a system is stable over a small, small disturbance. Here we shall discuss the stability of the system (4) around E_3 by constructing a suitable Lyapunov functional given in (15). Let $u_1(t)=x-x^*, u_2(t)=y-y^*$ and $u_3(t)=z-z^*$ then the system (4) can be written into the form as:

$$\frac{dA_1}{dt} = p_{11}u_1 + p_{12}u_2 - p_{13}u_3,$$
$$\frac{dA_2}{dt} = p_{21}u_1 + p_{22}u_2 - p_{23}u_3,$$
$$\frac{dA_3}{dt} = p_{31}u_1 + p_{32}u_2 + p_{33}u_3, \quad (14)$$