Chaos, Solitons and Fractals 114 (2018) 347–363

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Blind in a commutative world: Simple illustrations with functions and chaotic attractors

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ARTICLE INFO

Article history:
Received 21 June 2018
Revised 10 July 2018
Accepted 19 July 2018

Keywords:
Commutativity
Transfer functions
Continuous kernels
Chaotic attractors

ABSTRACT

The paper is devoted to investigate three different points including the importance, usefulness of the Bode diagram in calculus including classical and fractional on one hand. On the other hand to answer and disprove the statements made about fractional derivatives with continuous kernels. And finally to show researchers what we see and we do not see in a commutative world. To achieve this, we considered first the Caputo–Fabrizio derivative and used its Laplace transform to obtain a transfer function. We represented the Bode, Nichols, and the Nyquist diagrams of the corresponding transfer function. We in order to assess the effect of exponential decay filter used in Caputo–Fabrizio derivative, compare the transfer function associate to the Laplace transform of the classical derivative and that of Caputo–Fabrizio, we obtained surprisingly a great revelation, the Caputo–Fabrizio kernel provide better information than first derivative according to the diagram. In this case, we concluded that, it was not appropriate to study the Bode diagram of transfer function of Caputo–Fabrizio derivative rather, it is mathematically and practically correct to see the effect of the kernel on the first derivative as it is well-established mathematical operators. The Caputo–Fabrizio kernel Bode diagram shows that, the kernel is low past filter which is very good in signal point of view. We consider the Mittag–Leffler kernel and its corresponding Laplace transform and find out that due to the fractional order, the corresponding transfer function does not exist therefore the Bode diagram cannot be presented as there is no so far a mathematical formula that help to find transfer function of such nature. It is therefore an opened problem, how can we construct exactly a transfer function with the following term $\frac{(iw)^{\alpha}}{(iw)^{\alpha}+b}$ for instance? We proved that fractional derivative with continous kernel are best to model real world problems, as they do not inforce a non-singular model to become singular due to the singularity of the kernel. We show that, by considering initial time to be slightly above the origin then the Riemann–Liouville and Caputo-power derivatives are fractional derivatives with continuous kernel. We considered some interesting chaotic models and presented their numerical solutions in different ways to show what we see or do not see if a commutative world. To end, we presented the terms to be followed to provide a new fractional derivative.

1. Introduction

The concept of fractional differentiation, although 323 years in 2018, has really witnessed a great attention and consideration in the last 55 years more precisely in 1967 its first modification was done under the framework of Caputo investigation [1,2]. In order to include normal initial conditions while using the Laplace transform, Caputo was the first researcher to realize that by transforming a fractional differential operator from derivative of a convolution, to a convolution of a derivative with power, one will obtain normal initial condition rather than initial condition with no physical meaning and difficult to compute, this was a first time new contribution on the Riemann–Liouville operator was suggested [1–4]. These two fractional differential operators have been applied to model real world problems in many fields including in science, technology and engineering [5–10]. Thus, due to the successful results obtained while applying these mathematics tools in many fields, researchers working in technology, engineering and other fields of science than mathematics have also being trying to point out their viewpoints. Fractional differential and integral operators although being mathematics tools created by mathematicians have now become a centre of interest of every researcher. Very recently, it comes to Caputo and Fabrizio attention that, the commonly used fractional differential operators have some limitations and also produced misleading results while modelling real world problems, in particular these senior researchers were concerned with singularity of the power-law based fractional derivatives. They stressed on the fact that several physical occurrences

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https://doi.org/10.1016/j.chaos.2018.07.022
0960-0779/

are non-singular, their argument is very valid as using singular operators to model non-singular physical occurrence is mathematically and practically misleading. To solve this problem, they recalled the initial attempt of formulating a fractional differential operator as suggested by Liouville in 1832, where the exponential function was used as kernel. They chose a special exponential decay function with the property that when the fractional order tends to 1 we recover the Delta Dirac function or distribution in order to unable the recovery of a first derivative. Without any doubt the exponential function is the generator of an evolution partial differential equation with classical derivative, it is also the Eigen function of the decay differential ordinary equation. With these remarks in mind, some researchers suggested that, the kernel was not non-local, but they failed to specify that, the non-locality was that of time, this remark was absolutely correct, however, the kernel has another type of non-locality as was explained by Caputo and Fabrizio in their recent work [11,12]. To include the time non-locality into mathematical formulation of fractional differential operator with non-singular kernel, Atangana and Baleanu suggested the generalized Mittag–Leffler function as a new kernel. Both differential have been used with great success in many field of science, technology and engineering in the last two years. However, some researchers from other fields suggested some criteria to filter the flow of new differential operators. They pointed out the index-law and the observation of Bode diagram. Some researchers even went far to say these new operators did not add much in the field of fractional calculus, of course these statements were disproved in some research papers by researchers with strong background in applied mathematics and applications [12]. The restriction and limitations of the index law were presented in several papers, in fact, Nieto et al proved mathematically that, the index law was not possible in fractional calculus on one hand. On the other hand using the evolution equation approach Atangana proved with great success the failure of the index law in modelling real world problems [15–19,29]. In a discussion that took place via research gate where different opinions were suggested, Hristov suggested that such imposition of classical law fractional calculus was obstruction to the development of science and this could lead to the destruction of the world as science need to evolve to capture more complexities of nature. Thabet believed that "in everything which is mathematically true and I am sure it will be useful for researchers in different fields. If an engineer or biologist or an economist etc. Does not like certain mathematical concept for his specific model then he may leave it and apply another. He does not have the right to close the door about that concept since it might be interesting for others when they apply for other different models at least! On the other hand it is logical and natural to lose some nice properties when we generalize certain mathematical concept! But of course generalization is a great job in mathematics since it enables us to for example to work in a more general function space theoretically and hopefully to be applied for more classes of real world applications." Bilal Riaz believes that, "young researchers need new ideas to develop the field, as other fields were developed for instance Algebra where commutative algebra and non-commutative were developed". Zakia suggested that, investigation of fractional differential operators beyond the index law "is a great contribution in the field of fractional calculus and also it has opened the way forward to new results and applications to all field of science, technology and engineering. These results are similar to those of non-commutative geometry that leaded to field medal." All together, the idea of imposing the index-law of classical mechanic was not accepted in fractional calculus. In fact we should recall that, the initial conception of a differential operator was laws free until they discovered that, this operator was able to satisfy some properties. In this paper, we present a critical analysis of these two concerned including the commutativity and the Bode diagram suggested by some engineers and also we present a critical analysis on the work about continuous kernels fractional derivatives.

2. How commutative is your world?

The index-law imposed in fractional calculus falls under the commutativity. It is specific case of commutative operators. In this section, we present some trivial but yet very informative results that will lead us understand some limitation of the general concept of commutativity. We provide this illustration using the concept of functions. Indeed if we choice two different function says f, g, then we consider the composite operator, already we know that $f^\circ g \neq g^\circ f$, however, in a commutative world we will like to that the following $(f^\circ g)(x) = (g^\circ f)(x)$. Now the real question is following what do we gain and what do we lost? Another interesting question will be what do we see and we do not see?

If both functions are not invertible then the only conclusion is that $f = g$, however if both functions are invertible that mean $f = g^{-1}$. So the commutative world will only be able to represent these classes of functions. This is what we can capture in a commutative world. But the world of non-commutativity contains, commutative elements and then those elements that do not obey the commutative law. With this in mind, in a commutative world, we cannot have cover the set of functions $(f^\circ g)(x) \neq (g^\circ f)(x)$, which are more abundant in nature than those satisfying this law. So we call for instance W a set of functions, the man living in a commutative world will only see some and very few functions, more precisely those functions invertible, but will be blind to see non-invertible functions, whereas the man living in non-commutative world will see all the functions, including those not invertible. Let us see the representation of this illustration using some well-known functions. Let us consider the function (f, g) where $(sinx, x^2)$, we present the graphical representation of $(f^\circ g)$, $(g^\circ f)$ and $(f^\circ g - g^\circ f)$, since in a commutative world we can only have either $(f^\circ g)$ or $(g^\circ f)$, thus what we cannot see in commutative world will be $(f^\circ g - g^\circ f)$ (Figs. 1–3).

The above three figures show that, a man living in a commutative world can only perceive and what he cannot.

3. Misleading statements about continuous kernels

There exist in nature many type of singularities. In mathematics, singularities are points where mathematical object are not defined or not well behaved. In mathematical framework, we have singularities in geometry and also in complex analysis. The term appears in natural science and also technology. For instance, fractional differential operators based on power law are singular as we could have a blow up around the point zero. Due to this singularity, or discontinuity, some authors have suggested alternatives non-singular kernel including exponential decay law and the Generalized Mittag–Leffler function with delta Dirac property when the fractional order tends to 1. The new derivatives have been applied in several fields including science, technology and engineering. Several authors obtained surprising results [13–23]. This section aims to disprove the statements made [24] regarding non-singular or continuous kernels.

3.1. Analysis

Let us assume that, the analysis presented in [24] is true, then let us consider the Riemann–Liouville–Caputo derivative defined as:

$$ {}_{\in}^{C}D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(1-\alpha)} \int_{\in}^{t} (t-y)^{-\alpha} \frac{df(y)}{dy} dy, $$
$$ \in \text{ is extremely small enough} $$