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Computation of the largest positive Lyapunov exponent using rounding mode and recursive least square algorithm

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A B S T R A C T

It has been shown that natural interval extensions (NIE) can be used to calculate the largest positive Lyapunov exponent (LLE). However, the elaboration of NIE are not always possible for some dynamical systems, such as those modelled by simple equations or by Simulink-type blocks. In this paper, we use rounding mode of floating-point numbers to compute the LLE. We have exhibited how to produce two pseudo-orbits by means of different rounding modes; these pseudo-orbits are used to calculate the Lower Bound Error (LBE). The LLE is the slope of the line gotten from the logarithm of the LBE, which is estimated by means of a recursive least square algorithm (RLS). The main contribution of this paper is to develop a procedure to compute the LLE based on the LBE without using the NIE. Additionally, with the aid of RLS the number of required points has been decreased. Eight numerical examples are given to show the effectiveness of the proposed technique.

1. Introduction

It is generally accepted that the largest positive Lyapunov exponent (LLE) is one of the best approaches to detect the presence of chaos in a dynamical system [1–7]. Lyapunov exponents measure the average divergence or convergence of nearby trajectories along certain directions in state space. In chaotic systems, the states of two copies of the same system separate exponentially with time despite very similar initial conditions [8,9]. Several numerical methods to estimate LLE have been proposed since the work by Oseledec [10]. In general, Lyapunov exponents are computed by tracing the exponential divergence of close trajectories. This divergence is explored in [11] to calculate the LLE, although in [12] it is pointed out that such a method is not very robust and difficult to apply. To overwhelm this problem, Rosenstein et al. [1] and Kantz [12] have proposed a different strategy, in which the time dependence of distances between nearby trajectories is recorded explicitly to select the appropriate length scale and range of times from the output [2]. Examples to compute the LLE can be seen in [1,3,6,7,11–14,14–25], just to cite a few.

The relevance of the measure of the LLE and the observation of that two copies of the same system separate exponentially does not rely only on the characterization of the system is chaotic or not. Perc and Marhl [26] have developed a technique in which this featured is exploited to detect and control unstable periodic orbits. It is also important to state that the determination of LLE has been applied with success to acquire important insights into system dynamics [23–25,27]. Recently, Mendes and Nepomuceno [2] have presented a simple algorithm to estimate the LLE. The approach is based on the concept of the lower bound error (LBE) first introduced in [28] and further developed in [29]. To estimate the LLE, the system, either discrete or continuous, is simulated using two different natural interval extensions (NIE), which are the foundation used to calculate the LBE. Although, the method proposed in [2] brings some interesting developments, either for its simplicity and robustness or for the smaller amount of required data, it presents at least one downside, which is the need to elaborate NIE [30]. In a first instance, this seems to be an easy step, but soon we have realised that there are many cases in which NIE are not easily derived. For example, let the quadratic map [31] given by

$$x_{n+1} = 2 - x_n^2. \quad (1)$$

This map is in a very simplified form, which does not allow any change of sequence in the arithmetic operation to produce a different NIE. Besides that, there are dynamical systems, modelled by neural networks, such as in [32], which equations are not easily manipulated. We may also mention systems modelled by blocks, such as Simulink [33], which equations are not explicitly available. Thus, to overcome this limitation, we have found that two differ-

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ent rounding modes present similar effects to those produced by two NIE. Therefore, rounding mode has been applied instead of using NIE to calculate the LBE, and consequently the LLE. According to IEEE 754-2008 standard, the rounding mode indicates how the least significant returned digit of a rounded result is to be calculated [34–36], this can be simply obtained with an internal Matlab function [37] or in C^{++} [38]. From this point, this paper follows the steps presented in [2], where the LLE is obtained by a simple least square fit to the line of the natural logarithm of LBE, just about from the beginning of simulation up to the instant when the LBE stops increasing. We have also improved this stage replacing the least square by the recursive least square algorithm (RLS) [39]. This brings two main advantages: reduction of the number of points and automation of the process, as we do not need to set up beginning and end points of LBE range to calculate the slope, and thus the LLE. As in [1] the natural logarithm is adopted here. The method is applied successfully to eight numerical examples. Firstly, the same examples used in [1]: Logistic [40], Hénon [41], Lorenz [42], and Rössler equations [43] have been considered. We also included other four cases, namely: Sine Map [44], Tent Map [45], Mackey-Glass [46], and a Simulink version of Rössler adapted from Aseeri [47]. We have also investigated the results of the proposed method to calculate the LLE for a periodic dynamical system, which has obviously delivered a non-positive value.

Algorithm 1 Pseudo-code of the LLE calculation using Matlab, where mod1 and mod2 are two different rounding modes and RLS is the recursive least square algorithm according Eq. (6).

input Parameters, initial conditions, tol
Stop ← False
while Stop **do**
  |system_dependent('setround',mod1)|
  $\hat{x}_{a,n+1} \leftarrow f(\hat{x}_{a,n})$
  |system_dependent('setround',mod2)|
  $\hat{x}_{b,n+1} \leftarrow f(\hat{x}_{b,n})$
  |system_dependent('setround',0.5)|
  $\ell_{\Omega,n+1} \leftarrow (|\hat{x}_{a,n+1} - \hat{x}_{b,n+1}|)/2$
  $\lambda_{n+1} \leftarrow \mathrm{RLS}(\ell_{\Omega,n+1})$
  $\lambda_{5+} \leftarrow \max\{\lambda_{n+1}, \lambda_n, \cdots, \lambda_{n-3}\}$
  $\lambda_{5-} \leftarrow \min\{\lambda_{n+1}, \lambda_n, \cdots, \lambda_{n-3}\}$
  $\lambda_m \leftarrow \mathrm{mean}\{\lambda_{n+1}, \lambda_n, \cdots, \lambda_{n-3}\}$
  if $\frac{|\lambda_{5+} - \lambda_{5-}|}{|\lambda_m|} < tol$ **then**
    Stop ← True
  end if
end while

The remainder of the paper is organised as follows. Section 2 provides preliminary concepts about LBE. The main results are developed in Section 3. Section 4 is devoted to illustrate the results and final remarks are given in Section 5.

2. The lower bound error

In this section, some definitions on recursive functions, NIE and pseudo-orbits are shown. After that, the theorem of LBE is presented [28]. Let $n \in \mathbb{N}$, a metric space $M \subset \mathbb{R}$, the relation

$$x_{n+1} = f(x_n), \tag{2}$$

where f: $M \to M$, is a recursive function or a map of a state space into itself and x_n denotes the state at the discrete time n. The sequence $\{x_n\}$ obtained by iterating Eq. (2) starting from an initial condition x_0 is called the orbit of x_0 [48]. Let f be a function of real variable x. Moore and Moore [49] present the following definition.

Table 1
Chaotic systems investigated in this paper. The Rössler has also been modelled using Simulink, as described in Fig. 1. The sampling time is denoted by $\Delta t(s)$. The initial condition is arbitrarily adopted but fixed for the two rounding modes.

System	Equations	Parameters	$\Delta t(s)$	Initial Condition
Logistic	$x_{n+1} = \mu x_n(1 - x_n)$	$\mu = 4.0$	1	$x_0 = 2/3$
Hénon	$x_{n+1} = 1 - ax_n^2 + y_n$	$a = 1.4$	1	$x_0 = 0.3$
	$y_{n+1} = bx_n$	$b = 0.3$		$y_0 = 0.3$
Sine Map	$x_{n+1} = ax_n - bx_n^3$	$a = 2.6868$	1	$x_0 = 0.1$
		$b = 0.2462$		
Tent Map	$x_{n+1} = r\min\{x_n, 1\text{-}x_n\}$	r=1.99	1	$x_0 = 0.6$
Lorenz	$\dot{x} = \sigma(y - x)$	$\sigma = 16.0$	0.01	$x(0) = 1$
	$\dot{y} = x(\rho - z) - y$	$\rho = 45.92$		$y(0) = 0.5$
	$\dot{z} = xy - \beta z$	$\beta = 4.0$		$z(0) = 0.9$
Rössler	$\dot{x} = -y - z$	$a = 0.15$	0.10	$x(0) = -1$
	$\dot{y} = x + ay$	$b = 0.20$		$y(0) = 1$
	$\dot{z} = b + z(x - c)$	$c = 10.0$		$z(0) = 1$
Mackey-Glass	$\dot{x} = \frac{ax_\tau}{1 - x_\tau^c} - bx$	$a = 0.2,\ b = 0.1$	0.3	$x(0) = 0.3$
		$c = 10,\ \tau = 30$		

Table 2
Computation of the LLE (λ) given in natural logarithm. The last column presents the number needed iterates to calculate λ. The expected values are obtained in references indicated in the third column.

System	Literature λ	[Ref.]	Calculated λ	Iterates
Logistic	0.693	[4]	0.711	35
Hénon	0.418	[11]	0.408	89
Sine Map	0.773	[44]	0.794	26
Tent Map	0.688	[45]	0.684	16
Lorenz	1.500	[11]	1.390	2496
Rössler	0.092	[11]	0.092	1413
Rössler (Simulink)	0.092	[11]	0.092	1090
Mackey-Glass	0.0074	[18]	0.0069	10,178

Definition 2.1. A *natural interval extension* (NIE) of f is an interval valued function F of an interval variable X, with the property

$$F(x) = f(x) \quad \text{for real arguments}, \tag{3}$$

where by an interval we mean a closed set of real numbers $x \in \mathbb{R}$ such that $X = [\underline{X}, \bar{X}] = \{x : \underline{X} \le x \le \bar{X}\}$.

Connected to a map an orbit may be defined as follows:

Definition 2.2. An orbit is a sequence of values of a map, represented by $\{x_n\} = [x_0, x_1, \ldots, x_n]$.

Definition 2.3. Let $i \in \mathbb{N}$ represents a *pseudo-orbit, which is defined by an initial condition, a natural interval extension of f, some specific hardware, software, numerical precision standard and discretization scheme.* A pseudo-orbit approximates an orbit and can be represented as

$$\{\hat{x}_{i,n}\} = [\hat{x}_{i,0}, \hat{x}_{i,1}, \ldots, \hat{x}_{i,n}],$$

such that

$$|x_n - \hat{x}_{i,n}| \le \gamma_{i,n}, \tag{4}$$

where $\gamma_{i,n} \in \mathbb{R}$ is a bound of the error and $\gamma_{i,\,n} \ge 0$.

Nepomuceno et al. [29] have shown that two pseudo-orbits derived from associative multiplication property presents the same error bounds. These extensions have been called in such work as *arithmetic interval extension*. The lower bound error theorem has been proved in [29]:

Theorem 2.4. *Let $\{\hat{x}_{a,n}\}$ and $\{\hat{x}_{b,n}\}$ be two pseudo-orbits derived from two arithmetic interval extensions. Let $\ell_{\Omega,n} = |\hat{x}_{a,n} - \hat{x}_{b,n}|/2$ be the lower bound error associated to the set of pseudo-orbits $\Omega = [\{\hat{x}_{a,n}\}, \{\hat{x}_{b,n}\}]$ of a map, then $\gamma_{a,n} = \gamma_{b,n} \ge \ell_{\Omega,n}$.*