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Study on the mild solution of Sobolev type Hilfer fractional evolution equations with boundary conditions

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ABSTRACT

This paper is concerned with the fractional differential equations of Sobolev type with boundary conditions in a Banach space. With the help of properties of Hilfer fractional calculus, the theory of propagation family as well as the theory of the measure of noncompactness and the fixed point methods, we obtain the existence results of mild solutions for Sobolev type fractional evolution differential equations involving Hilfer fractional derivative. Finally, two examples are presented to illustrate the main result.

1. Introduction

In the last decades, fractional calculus and fractional differential equations have attracted much attention, we refer to [1,2,7,8,8,10,13,14,19,40–47] and references therein. It is found that many phenomena can be modeled with the help of fractional derivatives or integrals, such as fractional Brownian motion [5], anomalous diffusion [12,15], etc. Fractional differential equations have been applied to various fields successfully, for example, physics, engineering, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [1,3–6]. It is a development in the theory and application of fractional differential equations with Riemann–Liouville fractional derivative or the Caputo fractional derivative, see [25–32] and reference therein.

In recent years, Hilfer fractional differential equations have received much attention. Hilfer [18–20] proposed a generalized Riemann–Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann–Liouville fractional derivative and Caputo fractional derivative. It seems that Hilferet al. [1,21] have initially proposed linear differential equations with the new fractional operator: Hilfer fractional derivative and applied operational calculus to solve such simple fractional differential equations. Thereafter, Furati et al. [8] discussed the existence and uniqueness for the general problem

$$\begin{cases} D_{a+}^{\nu,\mu}u(t) = f(t,u(t)), & 0\le\nu\le 1, 0<\mu<1,\ t>a \\ I_{a+}^{1-\gamma}u(a+) = c, & c>0,\ \mu\le\gamma=\mu+\nu-\mu\nu<1, \end{cases}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative. Next, Wang and Zhang [33] extend the above initial condition to nonlocal boundary value problem of the form

$$\begin{cases} D_{a+}^{\nu,\mu}u(t) = f(t,u(t)), & 0\le\nu\le 1, 0<\mu<1,\ t\in(a,b], \\ I_{a+}^{1-\gamma}u(0) = \sum_{i=1}^{m}\lambda_i u(\tau_i), & \mu\le\gamma=\mu+\nu-\mu\nu<1, \quad \tau_i\in(a,b]. \end{cases}$$

In [34], Gao and Yu studied Hilfer integral boundary value problems for the following relaxation fractional differential equations:

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) = cu(t)+f(t,u(t)), \quad c>0, \quad 0\le\nu \\ \qquad \le 1, 0<\mu<1,\ t\in(0,b], \\ I_{0+}^{1-\gamma}u(0+) = \sum_{i=1}^{m}\lambda_i I_{0+}^{1-\gamma}u(\tau_i), \quad \mu\le\gamma=\mu+\nu-\mu\nu \\ \qquad <1, \quad \tau_i\in(0,b). \end{cases}$$

involving Hilfer fractional derivatives, $0\le\nu\le 1$, $0<\mu<1$, by using Mittag–Leffler functions.

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In particular, Gu and Trujillo [9] investigated a class of evolution equations

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) = Au(t) + f(t, u(t)), \quad t \in (0, b], \\ I_{0+}^{(1-\nu)(1-\mu)}u(0) = u_0, \end{cases}$$

with Hilfer fractional derivatives, by Laplace transform and density function, they firstly give the mild solution definition.

On the other hand, Sobolev (or also called degenerate) type equation appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and so on [35–38,48]. The existence result of mild solutions of fractional integrodifferential equations of Sobolev-type with nonlocal condition in a separable Banach space is studied by using the theory of propagation family as well as the theory of the measures of noncompactness and the condensing maps [10]. Recently, we use the fixed point theorems combined with the theory of propagation family to discuss the existence of mild solutions for nonlinear fractional non-autonomous evolution equations of Sobolev type with delay of the form

$$\begin{cases} D_{0+}^{\nu,\mu}(Bu(t)) = Au(t) + Bf(t, u(\tau_1(t)), \dots, u(\tau_m(t))), \quad t \in J, \\ I_{0+}^{(1-\nu)(1-\mu)}Bu(0) = Bu_0, \end{cases}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative which will be given in next section, $0 \le \nu \le 1$, $0 < \mu < 1$, the state $u(\cdot)$ takes value in a Banach space E. $J = [0, b](b > 0), J' = (0, b]$. This work is based on the theory of propagation family $\{W(t)\}_{t\ge 0}$ introduced by Liang and Xiao [12], and measure of noncompactness which ensure us to not assume the nonlinear term f satisfies a Lipschitz type condition, for more detail see [21].

To the best of our knowledge, there has no results about Hilfer fractional evolution differential equations of Sobolev type with boundary conditions. Motivated by the above discussion, in this paper, we use the fixed point theorems combined with the theory of propagation family to discuss the existence of mild solutions for Existence results for Hilfer fractional evolution differential equations of Sobolev type with boundary conditions of the form

$$\begin{cases} D_{0+}^{\nu,\mu}(Bu(t)) = Au(t) + Bf\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))ds\Big), \\ \qquad 0 \le \nu \le 1, 0 < \mu < 1, \ t \in J, \\ I_{0+}^{1-\gamma}Bu(0) = \sum\limits_{i=1}^{m} \lambda_i I_{0+}^{1-\gamma}Bu(\tau_i), \quad \mu \le \gamma = \mu + \nu \\ \qquad -\mu\nu < 1, \quad \tau_i \in (0, b], \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative of order μ and type ν, which is a interpolator between Riemann–Liouville and Caputo fractional derivatives. The operator $D_{0+}^{\nu,\mu}$ is a generalized of Riemann–Liouville fractional derivative operator introduced by Hifter in [18–20], the state $u(\cdot)$ takes value in a Banach space E. $J = [0, b](b > 0), J' = (0, b]$. A and B are closed (unbounded) linear operator with domains contained in E, the pair (A, B) generate a propagation family $\{W(t)\}_{t\ge 0}$. f: $[0, b] \times E \times E \to D(B) \subset E$, g: $C([0, b], E) \to D(B) \subset E$ are given functions to be specified later, $\rho : \Delta \to R, h : \Delta \times E \to E(\Delta = \{(t, s) \in [0, b] \times [0, b] : t \ge s\}), \tau_i, i = 1, 2, \dots, m$ are pre-fixed points satisfying $0 < \tau_1 \le \cdots \le \tau_m < b$ and λ_i are real numbers. we study (1.1) without assuming the existence of B^{-1} as a bounded operator as well as without any assumption on the relation between $D(A)$ and $D(B)$. This work is based on the theory of propagation family $\{W(t)\}_{t\ge 0}$ (an operator family generated by the operator pair (A, B), see Lemma 2.11) introduced in [12], and measure of noncompactness and the fixed point methods, we obtain the existence results of mild solutions, which in general is improved for the paper [35–38,48].

It is easily seen that problem (1.1) contains many important classes of Cauchy problems for differential equations. In the nondegenerate case ($B = I$), the problem with initial conditions has been studied in [9]. For an initiative of study in degenerate case, we recall the work [10,21]. As mentioned in [10], in the case when B has a bounded inverse B^{-1}, one can reformulate (1.1) as follows

$$\begin{cases} {}^C D_{0+}^{\alpha}(Bu(t)) = Au(t) + \widetilde{f}\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))ds\Big), \ t \in J, \\ Bu(0) = Bu_0, \end{cases} \tag{1.2}$$

since we have $f = B^{-1}(Bf) = B^{-1}\widetilde{f}$, $\widetilde{f}$ and f may have the same setting. The existence of mild solutions of fractional differential systems (for the Caputo fractional derivative) in the form of (1.2) has been studied by several authors by assuming that $D(B) \subset D(A)$, B is bijective and $B^{-1} : E \to D(B)$ is a compact operator. In this case, AB^{-1} is a bounded operator and generates a compact C_0-semigroup $T(t) = e^{AB^{-1}t}$, for $t \ge 0$, and there exist two characteristic solution operators $S_\alpha(t)$ and $T_\alpha(t)$ given by the following nice subordination formulas

$$S_\alpha(t) = \int_0^\infty B^{-1}\xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \tag{1.3}$$

$$T_\alpha(t) = \alpha\int_0^\infty B^{-1}\theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \tag{1.4}$$

where $t \ge 0$ and

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-(1+\frac{1}{\alpha})}\varpi(\theta^{-\frac{1}{\alpha}}),$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).$$

By using the representations (1.3) and (1.4) several and interesting results on the existence of mild solutions of system (1.2) have been obtained, for [35–38] and the references therein. Finally, we mention here that a different approach without the representations (1.3) and (1.4) was done in [10] by using the propagation family $\{W(t)\}_{t\ge 0}$. However, such an assumption is not required in this work.

The rest of this paper is organized as follows: In Section 2, we recall some basic known results and introduce some notations. In Section 3, we discuss the existence theorems of mild solutions for the problem (1.1). At last, two examples will be presented to illustrate the main results.

2. Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel. Throughout this work, we set $J = [0, b]$, $J' = (0, b]$, where $b > 0$ is a constant. Let E be a Banach space with the norm $\|\cdot\|$ and the pair (A, B) generate a propagation family $\{W(t)\}_{t\ge 0}$ (see Definition 2.6). We denote by $B(E)$ the Banach space of all bounded linear operators from E to E. For a closed and linear operator A: $D(A) \subset E \to E$, where $D(A)$ is the domain of A, we denote by $\rho(A)$ its resolvent set. We also denote by $C(J, E)$ the Banach space of all continuous E-value functions on interval J with the norm $\|u\| = \sup_{t\in J}\|u(t)\|$. Let

$$C_{1-\gamma}(J, E) = \Big\{u : J' \to E | t^{1-\gamma}u(t) \in C(J', E)\Big\}$$

with the norm $\|\cdot\|_{C_{1-\gamma}}$ defied by

$$\|u\|_{C_{1-\gamma}} = \sup_{0\le t\le b}\Big|t^{1-\gamma}u(t)\Big|.$$