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Randomly orthogonal factorizations with constraints in bipartite networks☆

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ABSTRACT

Many problems on computer science, chemistry, physics and network theory are related to factors, factorizations and orthogonal factorizations in graphs. For example, the telephone network design problems can be converted into maximum matchings of graphs; perfect matchings or 1-factors in graphs correspond to Kekulé structures in chemistry; the file transfer problems in computer networks can be modelled as $(0, f)$-factorizations in graphs; the designs of Latin squares and Room squares are related to orthogonal factorizations in graphs; the orthogonal (g, f)-colorings of graphs are related to orthogonal (g, f)-factorizations of graphs. In this paper, the orthogonal factorizations in graphs are discussed and we show that every bipartite $(0, mf-(m-1)r)$-graph G has a $(0, f)$-factorization randomly r-orthogonal to n vertex disjoint mr-subgraphs of G in certain conditions.

1. Introduction

Many real-world networks can conveniently be modelled by graphs or networks. Examples include a railroad network with nodes presenting railroad stations and links corresponding to railways between two stations, or a communication network with nodes presenting cities, and links presenting communication channels. In particular, a wide variety of systems can be described using complex networks. Such systems include: the World Wide Web, which is a virtual network of Web pages connected by hyperlinks; the cell, where we model the chemicals by nodes and their interactions by edges; and the food chain webs, the networks by which human diseases spread, human collaboration networks etc [1]. Factors and orthogonal factorizations in networks or graphs have attracted a great deal of attention [2–10] due to their applications in network design, circuit layout, combinatorial design, and so on. For example, a Room square of order $2n$ is related to the orthogonal 1-factorization of a complete graph K_{2n}, which was first posed by Horton [12]. Euler [11] first found that a pair of orthogonal Latin squares of order n is equivalent to two orthogonal 1-factorizations of a complete bipartite graph $K_{n,n}$. Many other applications in this field can be found in a current survey [13]. It is well known that a network can be represented by a graph. Nodes of the network correspond to vertices of the graph, and links between the nodes in the network correspond to edges of the graph. Henceforth we use the term graph instead of network.

The graphs considered in this paper will be finite undirected graphs which neither loops nor multiple edges. Let $G=(V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. For arbitrary $x \in V(G)$, the degree of x in G is defined as the number of edges which are adjacent to x and denoted by $d_G(x)$. Let g, f: $V(G) \to Z$ be two functions such that $f(x) \geq g(x) \geq 0$ for each $x \in V(G)$. A spanning subgraph F of G with $g(x) \leq d_F(x) \leq f(x)$ for arbitrary $x \in V(G)$ is defined as a (g, f)-factor of G. Especially, G is defined as a (g, f)-graph if G itself is a (g, f)-factor. A (g, f)-factorization $\mathcal{F}=\{\mathcal{F}_\infty, \mathcal{F}_\in, \ldots, \mathcal{F}_\Updownarrow\}$ of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \ldots, F_m$. A subgraph H of a graph G is said to be an m-subgraph if H admits m edges in total. For a (g, f)-factorization $\mathcal{F}=\{\mathcal{F}_\infty, \mathcal{F}_\in, \ldots, \mathcal{F}_\Updownarrow\}$ of G and an mr-subgraph H of G, $\mathcal{F}$ is said to be r-orthogonal to H if $|E(H) \cap E(F_i)| = r$, $1 \leq i \leq m$. If for any partition $\{A_1, A_2, \ldots, A_m\}$ of $E(H)$ with $|A_i| = r$, G admits a (g, f)-factorization with $A_i \subseteq E(F_i)$, $1 \leq i \leq m$, then we say that G admits (g, f)-factorizations randomly r-orthogonal to H. It is trivial that randomly 1-orthogonal is

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equivalent to 1-orthogonal and 1-orthogonal is also said to be orthogonal. A graph denoted by $G=(X,Y,E(G))$ is called a bipartite graph with bipartition {X, Y} and edge set $E(G)$.

Alspach et al. [13] presented the following question: For a given subgraph H of G, does there exist a factorization $\mathcal{F}$ of G with some fixed type orthogonal to H?

Li and Liu [4] showed that every $(mg+m-1, mf-m+1)$-graph G admits a (g, f)-factorization orthogonal to arbitrary given m-subgraph. Li et al. [5] justified that for every $(mg+k, mf-k)$-graph G and its any given k-subgraph H, there exists a subgraph R such that R has a (g, f)-factorization orthogonal to H. Liu and Long [14] verified that every $(mg+m-1, mf-m+1)$-graph G has a (g, f)-factorization randomly r-orthogonal to arbitrary given mr-subgraph H. Liu and Dong [15] proved that every bipartite $(mg+m-1, mf-m+1)$-graph G admits a (g, f)-factorization orthogonal to k vertex disjoint m-subgraph. Liu and Zhu [16] justified that every bipartite $(mg+m-1, mf-m+1)$-graph admits (g, f)-factorizations randomly r-orthogonal to any given mr-subgraph. Zhou [17] studied the orthogonal (0, f)-factorizations in bipartite $(0, mf-(m-1)r)$-graphs.

Theorem 1. *(Zhou [17]). Let G be a bipartite $(0, mf-(m-1)r)$-graph, and let f be an integer-valued function defined on $V(G)$ with $f(x)\geq 2r$ for every $x\in V(G)$, and let H be an mr-subgraph of G. Then G admits a (0, f)-factorization randomly r-orthogonal to H.*

In this paper, the result in [17] is extended, and it is verified that every bipartite $(0, mf-(m-1)r)$-graph G admits a (0, f)-factorization randomly r-orthogonal to n vertex disjoint mr-subgraphs. Our main result will be shown in Section 3.

2. Preliminary lemmas

Let G be a graph, and let S and T be two disjoint vertex subsets of G. We use $E_G(S, T)$ to denote the set of edges with one end in S and the other in T, and write $e_G(S,T)=|E_G(S,T)|$. For $S\subset V(G)$, $G-S$ is the subgraph obtained from G by deleting the vertices in S together with the edges to which the vertices in S are incident, and $G[S]$ is the subgraph of G induced by S. For $E'\subset E(G)$, $G-E'$ is the subgraph obtained from G by deleting the edges in E', and $G[E']$ is the subgraph of G induced by E'. For any $X\subseteq V(G)$, we define $f(X)=\sum_{x\in X} f(x)$ for arbitrary function f defined on $V(G)$, and write $f(\emptyset)=0$.

The following necessary and sufficient condition for a bipartite graph to have a (g, f)-factor was obtained by Folkman and Fulkerson (see Theorem 6.8 in [18]).

Lemma 1. *Let $G=(X,Y,E(G))$ be a bipartite graph, and g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x)\leq f(x)$ for every $x\in V(G)$. Then G contains a (g, f)-factor if and only if*

$$\gamma_{1G}(S,T;g,f)=f(S)-g(T)+d_{G-S}(T)\geq 0$$

and

$$\gamma_{2G}(S,T;g,f)=f(T)-g(S)+d_{G-T}(S)\geq 0$$

for all $S\subseteq X$ and $T\subseteq Y$.

It is obvious that $d_{G-T}(S)=e_G(S,Y\setminus T)$ and $d_{G-S}(T)=e_G(T,X\setminus S)$. Let $S\subseteq X$ and $T\subseteq Y$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Define

$$E_{iS}=E_i\cap E_G(S,Y\setminus T),\qquad E_{iT}=E_i\cap E_G(T,X\setminus S)$$

for $i=1,2,$ and write

$$\alpha_S=|E_{1S}|,\quad \alpha_T=|E_{1T}|,\quad \beta_S=|E_{2S}|,\quad \beta_T=|E_{2T}|.$$

It is trivial that $\alpha_S\leq d_{G-T}(S),\ \alpha_T\leq d_{G-S}(T),\ \beta_S\leq d_{G-T}(S),\ \beta_T\leq d_{G-S}(T)$.

Liu and Zhu [16] obtained a necessary and sufficient condition for a bipartite graph to have a (g, f)-factor including E_1 and excluding E_2, which plays a crucial role for proving Theorem 2.

Lemma 2. *(Liu and Zhu [16]). Let $G=(X,Y,E(G))$ be a bipartite graph, and let g and f be two integer-valued functions defined on $V(G)$ satisfying $0\leq g(x)\leq f(x)$ for any $x\in V(G)$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G admits a (g, f)-factor F with $E_1\subseteq E(F)$ and $E_2\cap E(F)=\emptyset$ if and only if*

$$\gamma_{1G}(S,T;g,f)\geq \alpha_S+\beta_T$$

and

$$\gamma_{2G}(S,T;g,f)\geq \alpha_T+\beta_S$$

for all $S\subseteq X$ and $T\subseteq Y$.

In the following, we always assume that G is a bipartite $(0, mf-(m-1)r)$-graph, where m, r are two positive integers. Write

$$g(x)=\max\{0, d_G(x)-(m-1)f(x)+(m-2)r\}$$

for any $x\in V(G)$. In terms of the definition of $g(x)$, it is trivial that $0\leq g(x)\leq f(x)$ for all $x\in V(G)$. For any $x\in V(G)$, we write

$$\Delta_1(x)=\frac{1}{m}d_G(x)-g(x)$$

and

$$\Delta_2(x)=f(x)-\frac{1}{m}d_G(x).$$

Lemma 3. *(Zhou [17]). For any $S\subseteq X$ and $T\subseteq Y$, the following equalities hold:*

$$\gamma_{1G}(S,T;g,f)=\Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)$$

and

$$\gamma_{2G}(S,T;g,f)=\Delta_1(S)+\Delta_2(T)+\frac{m-1}{m}d_{G-T}(S)+\frac{1}{m}d_{G-S}(T).$$

3. Main result and its proof

The study of 1-factorizations is motivated by other combinatorial applications such as scheduling tournaments. Here, the schedule of games played at the same time can be seen to form a 1-factor of the underlying complete graph. If a round robin tournament for $2n$ teams is to be played in the minimum number of sessions, we require a 1-factorization of K_{2n}, together with an ordering of the factors (this ordering is sometimes irrelevant). If there are $2n-1$ teams, the relevant structure is a near-one-factorization of K_{2n-l}. In each case the (ordered) factorization is called the schedule of the tournament [19–22]. Other applications of 1-factorizations include block designs, 3-designs and Steiner systems [19,23–25]. The file transfer problems in computer networks can be modelled as (0, f)-factorizations of a graph [26].

Orthogonal factorizations have a wide range of applications in Room squares, Latin squares, colorings, and so on. Define a Room k-design of order n to be a k-dimensional array of cells, each either being empty or containing a pair chosen from a given set of n elements, such that, if one takes a two-dimensional projection of the array (that is if one chooses any two coordinates and fills a square array by placing into cell (i, j) the union of all cells with (i, j) in the two chosen coordinates), the result is always a Room square [12]. The following result was verified by Horton [12].

Theorem 2. *(Horton [12]). A Room k-design of order n is equivalent to k pairwise orthogonal 1-factorizations of a complete graph K_n.*

A Latin square of side n is an $n\times n$ array with entries from $\{1,2,\dots,n\}$, in which every row and every column contains each