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Twin birds inside and outside the cage

Sajad Jafari [a,*], Soroush Dehghan [a], Guanrong Chen [b], Sifeu Takougang Kingni [c], Karthikeyan Rajagopal [d]

[a] Biomedical Engineering Department, Amirkabir University of Technology, Tehran 15875-4413, Iran
[b] Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong SAR, China
[c] Department of Mechanical and Electrical Engineering, Faculty of Mines and Petroleum Industries, University of Maroua, P.O. Box 46, Maroua, Cameroon
[d] Centre for Non-Linear Dynamics, Defense University, Ethiopia

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ABSTRACT

This paper introduces a chaotic system in the spherical coordinates which, when expressed in the Cartesian coordinate system, has a chaotic attractor located in an impassable sphere like a bird in the cage. It also has a coexisting attractor outside that sphere. Basic dynamical properties of this system are investigated and its FPGA realization is demonstrated.

1. Introduction

It has been widely recognized that mathematically simple systems of nonlinear differential equations can exhibit chaos [1]. In recent years, finding special chaotic systems with unusual features has been a hot topic. As examples we can mention chaotic systems with multi-stabilities [2–4], with extreme multi-stabilities [5–9], with megastability [10,11], and chaotic systems with multi-scroll attractors [12–14], among others. However, the majority of such new systems are those with special distributions of equilibria. Systems with non-hyperbolic equilibria [15], with no equilibria [16,17], with stable equilibria [18], with lines of equilibria [19], with curves of equilibria [20], with planes of equilibria [21], and with surfaces of equilibria [22,23], etc., are typical ones. Almost all of these systems belong to a new category of dynamical systems: systems with hidden attractors [24–26]. A hidden attractor is an attractor that has a basin of attraction not intersecting with any small neighborhood of any equilibrium point. Hidden attractors are important in engineering applications because they admit unexpected and potentially disastrous responses to perturbations, for instance, in structures like a bridge or an airplane wing [27].

This paper introduces a chaotic system, which has two coexisting chaotic attractor inside and outside of an impassable sphere. We are aware of no other similar system. To start with, it should be noted that this system is completely different from those presented in [22]. The surface of equilibria in [22] is something additive to the system and the topology of the strange attractor does not change even if the surface is removed. Moreover, even the simplest forms of such systems have at least one cubic term, but on the contrary the system described in this paper has only quadratic terms.

In the next section, the new system is introduced and its dynamical properties are investigated. In Section 3, this new system is implemented via FPGA design, showing its feasibility in potential engineering applications. Finally, discussion and conclusion are given in Section 4.

2. A chaotic system with trajectories confined in an impassable sphere

Consider the following system, given in the spherical coordinate system (Fig. 1):

$$\begin{aligned}\dot{\rho} &= 4\rho\varphi - 16\varphi \\ \dot{\theta} &= \rho^2 + 3\varphi - 8\rho + 15 \\ \dot{\varphi} &= -a\theta - \varphi\end{aligned} \tag{1}$$

The design of system (1) is inspired by those proposed in [28–30], and it has three equilibrium points: $E_1 = (3, 0, 0)$, $E_2 = (5, 0, 0)$ and $E_3 = (4, -1/3a, 1/3)$, where $a \neq 0$. The characteristic

* Corresponding author.
E-mail address: sajadjafari@aut.ac.ir (S. Jafari).

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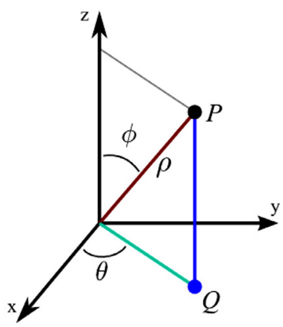

Fig. 1. Spherical coordinate system.

equation of the system Jacobian at $E^*(\rho^*, \theta^*, \varphi^*)$ is

$$\lambda^3 + (1 - 4\varphi^*)\lambda^2 + (3a - 4\varphi^*)\lambda + 4a\left[2(\rho^* - 4)^2 - 3\varphi^*\right] = 0 \tag{2}$$

For the equilibrium points $E_1 = (3, 0, 0)$ and $E_2 = (5, 0, 0)$, one has $\lambda^3 + \lambda^2 + 3a\lambda + 8a = 0$. According to the Routh-Hurwith criteria, the equilibrium point $E_{1,\,2}$ are unstable for $a > 0$. For the equilibrium point E_3, one has $\lambda^3 - \frac{\lambda^2}{3} + (3a - \frac{4}{3})\lambda - 4a = 0$, implying that this equilibrium is unstable for any value of a, again by the Routh-Hurwith criteria.

Bifurcation diagram and Lyapunov exponents of system (1) versus the parameter a are presented in Fig. 2, where the maximum

Table 1
Resources utilized by system (4).

Resource	Utilization	Available	Utilization %
LUT	723	32,600	2.22
FF	192	65,200	0.29
DSP	8	120	6.67
IO	97	210	46.19
BUFG	1	32	3.13

Lyapunov exponent is positive for $0 \leq a \leq 2.5$. The initial conditions are chosen inside the sphere.

It can be seen from Eq. (1) that, when the radius of a trajectory is 4 ($\rho = 4$), its velocity become zero ($\dot{\rho} = 0$).Thus, there will be no more changes in the radius therefore the trajectory remains on the surface of $\rho = 4$, which is a sphere with radius of 4 around the origin (corresponding to $x^2 + y^2 + z^2 = 4$ in the Cartesian coordinate system). Showing the trajectory in the Cartesian coordinate system via the transformation (3), one obtains Figs. 3 and 4.

$$\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned} \tag{3}$$

This system has another interesting property. There is a coexisting attractor outside the sphere. It can be seen in Fig. 5

Fig. 6 is the plot of basin of attraction of these 2 attractors. It shows that all the points inside the sphere ($\rho < 4$) go to the inner strange attractor. The points on the sphere ($\rho = 4$) remain on

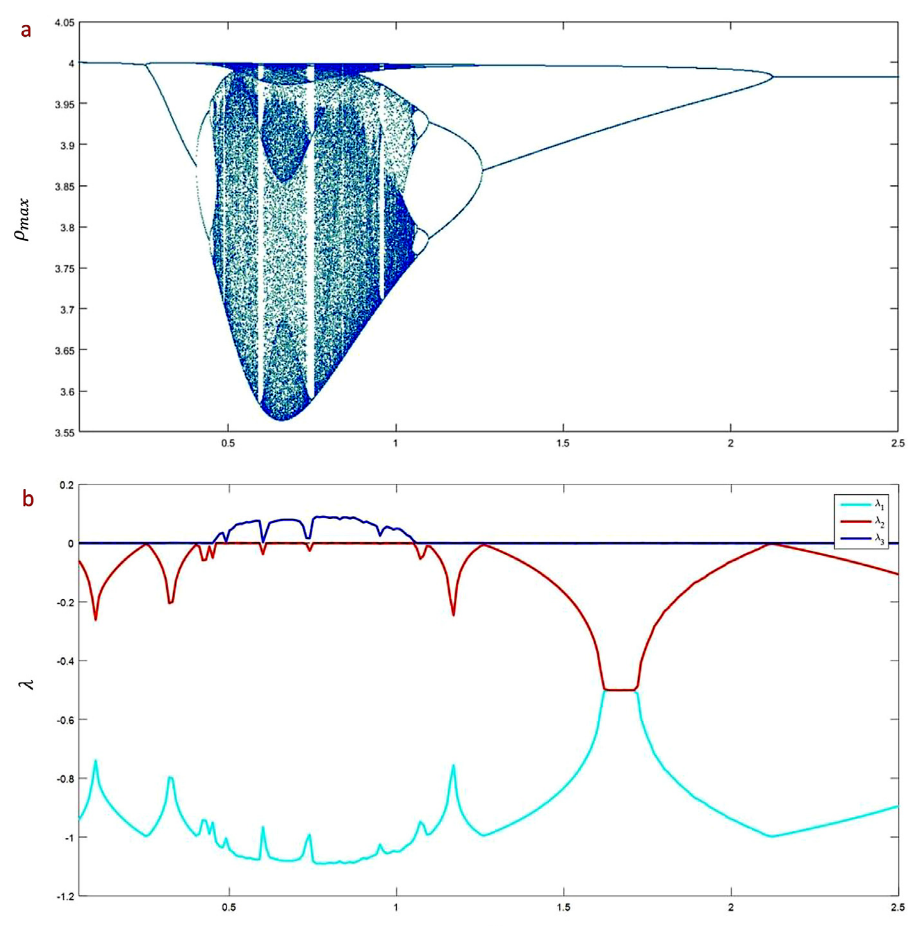

Fig. 2. a) Maximum values of "φ" versus parameter a in system (1), and b) The corresponding Laypunov exponents. The initial conditions are chosen inside the sphere.