Chaos, Solitons and Fractals 111 (2018) 18–34

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Limit cycles in small perturbations of a planar piecewise linear Hamiltonian system with a non-regular separation line☆

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ARTICLE INFO

Article history:
Received 5 November 2017
Revised 3 February 2018
Accepted 3 April 2018

MSC:
34C05
34C07
37G15

Keywords:
Limit cycle
Homoclinic loop
Melnikov function
Chebyshev system
Piecewise smooth system

ABSTRACT

We study Poincaré bifurcation for a planar piecewise near-Hamiltonian system with two regions separated by a non-regular separation line, which is formed by two rays starting at the origin and such that the angle between them is $\alpha \in (0, \pi)$. The unperturbed system is a piecewise linear system having a periodic annulus between the origin and a homoclinic loop around the origin for all $\alpha \in (0, \pi)$. We give an estimation of the maximal number of the limit cycles which bifurcate from the periodic annulus mentioned above under n-th degree polynomial perturbations. Compared with the results in [13], where a planar piecewise linear Hamiltonian system with a straight separation line was perturbed by n-th degree polynomials, one more limit cycle is found. Moreover, based on our Lemma 2.5 we improve the upper bounds on the maximal number of zeros of the first order Melnikov functions derived in [19].

1. Introduction and main results

In the past few decades many studies have been devoted to the investigation of limit cycles of planar piecewise smooth dynamical systems defined on two zones separated by a line, see e.g. [1–18, 21]. Commonly, the separation line of the two zones is straight. In this case using numerical simulations Huan and Yang [7] provided an example illustrating the existence of three limit cycles for piecewise linear systems. Rigorous proofs of the existence of these limit cycles were given in [8,9]. In [10] Chen and Du constructed a quadratic system, which can have nine small limit cycles bifurcated from a center. Perturbing quadratic systems with quadratic and cubic polynomial functions Xiong [11] and Tian and Yu [12] found six and ten limit cycles, respectively. For piecewise polynomial near-Hamiltonian systems of degree n, the number of limit cycles was studied in [13–15].

When the separation line is not a straight one, more than three limit cycles can be found for planar piecewise linear systems with two regions. For examples, by constructing some broken lines as the boundary between the two linear zones, Braga and Mello [16] proved that 4, 5, 6 or 7 limit cycles can appear. Two more concrete systems with piecewise continuous separation lines were given in [17], where Corollaries 3 and 5 confirm the following conjecture proposed in [16]:

For a given $n \in \mathbb{N}$ there is a piecewise linear system with two zones in the plane with exactly n limit cycles.

Recently, Cardin and Torregrosa [18] considered a planar piecewise linear system defined in two angular zones separated by

$$\Sigma_\alpha = \begin{cases} \{(x, y)|\ x \geq 0, y = 0\} \cup \{(x, y)|\ y = (\tan\alpha) \\ \quad x, y \geq 0\},\ \ \alpha \in (0, \pi) - \{\pi/2\}, \\ \{(x, y)|x \geq 0, y = 0\} \cup \{(x, y)|x = 0, y \geq 0\},\ \ \alpha = \pi/2. \end{cases} \tag{1.1}$$

Using higher order Melnikov function method the authors showed the existence of five limit cycles for such system up to a sixth order perturbation in ε.

☆ The first author is supported by the National Natural Science Foundation of China (No. 11671013). The second author is supported by the Slovenian Research Agency (program P1-0306, project N1-0063).

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https://doi.org/10.1016/j.chaos.2018.04.002
0960-0779/

To distinguish from the case of straight lines, if a separation line is piecewise smooth, we call it a non-regular one. It is pointed out in [16] that non-regularity in the separation lines plays an important role in studying of the number of limit cycles for piecewise systems. This motivates us to consider the following problems:

(a) For planar polynomial differential systems of degree n, what is the maximal number of limit cycles produced by perturbing a planar piecewise linear system with a period annuls around the origin, up to the first order in ε, if the piecewise period annulus is divided by $\Sigma_\alpha, \alpha \in (0, \pi)$?

(b) Does the maximal number of limit cycles depend on the angle $\alpha \in (0, \pi)$?

To study these problems, in this paper we first construct a planar piecewise near-Hamiltonian system of degree n, whose unperturbed system is piecewise linear and divided by $\Sigma_\alpha, \alpha \in (0, \pi)$, such that for any $\alpha \in (0, \pi)$ the unperturbed system has a family of periodic orbits between the origin and a homoclinic loop around the origin. In [13] the authors also considered a planar piecewise linear Hamiltonian system with two zones perturbed by nth-degree polynomials. The main difference between the unperturbed systems constructed in this paper and the one in [13] is the non-regularity in the separation lines, where in [13] it is a straight line and in this paper it is a non-regular one (Σ_α, $\alpha \in (0, \pi)$). Then, using the first order Melnikov function we investigate the maximal number of limit cycles bifurcating from the periodic annulus, obtaining one more limit cycle than in [13]. This shows again the importance of non-regularity in separation lines for piecewise systems when the number of bifurcated limit cycles is considered.

In order to give a formula of the first order Melnikov function for near-Hamiltonian systems with a separation line $\Sigma_\alpha, \alpha \in (0, \pi)$, we consider

$$\begin{cases} \dot{x} = H_y + \varepsilon p(x, y), \\ \dot{y} = -H_x + \varepsilon q(x, y), \end{cases} \tag{1.2}$$

where

$$H(x, y, \alpha) = \begin{cases} H^+(x, y, \alpha), & (x, y) \in \Sigma_\alpha^+, \\ H^-(x, y, \alpha), & (x, y) \in \Sigma_\alpha^-, \end{cases}$$

$$p(x, y) = \begin{cases} p^+(x, y), & (x, y) \in \Sigma_\alpha^+, \\ p^-(x, y), & (x, y) \in \Sigma_\alpha^-, \end{cases}$$

$$q(x, y) = \begin{cases} q^+(x, y), & (x, y) \in \Sigma_\alpha^+, \\ q^-(x, y), & (x, y) \in \Sigma_\alpha^-, \end{cases}$$

$H^\pm$, $p^\pm$ and $q^\pm$ are C^∞ functions, $\varepsilon > 0$ is small, $\Sigma_\alpha^\pm$ indicate the angular sections of angles α and $2\pi - \alpha$, $\alpha \in (0, \pi)$, respectively.

Let us make the following assumption to $(1.2)|_{\varepsilon=0}$.

Assumption A. For each $\alpha \in (0, \pi)$, suppose that $(1.2)|_{\varepsilon=0}$ has a family of periodic orbits $L_\alpha(h)$, $h \in J(\alpha)$, rotating clockwise around the origin, where $J(\alpha)$ denotes an open interval. Each closed orbit $L_\alpha(h)$ intersects Σ_α transversally at points $A(h, \alpha) = \big(a(h, \alpha), 0\big)$ and $A_1(a_1(h, \alpha), b_1(h, \alpha))$ such that for $h \in J(\alpha)$

$$H^+\big(A(h, \alpha), \alpha\big) = H^+\big(A_1(h, \alpha), \alpha\big) = h,$$

$$H^-\big(A(h, \alpha), \alpha\big) = H^-\big(A_1(h, \alpha), \alpha\big), \ \ a(h, \alpha) > 0, \ \ b_1(h, \alpha) > 0.$$

Let $L_\alpha(h) = L_\alpha^+(h) \cup L_\alpha^-(h)$, $L_\alpha^\pm(h)$ are defined by $H^+(x, y, \alpha) = h$, $(x, y) \in \Sigma_\alpha^+$ and $H^-(x, y, \alpha) = H^-(A_1(h, \alpha), \alpha)$, $(x, y) \in \Sigma_\alpha^-$, respectively. The phase portrait of system $(1.2)|_{\varepsilon=0}$ is shown in Fig. 1. Using methods of Theorem 2.2 in [20] and Theorem 1.1 in [21], it is easy to obtain the following Lemma.

Lemma 1.1. *Under Assumption A, for the first order Melnikov function of system (1.2), we have*

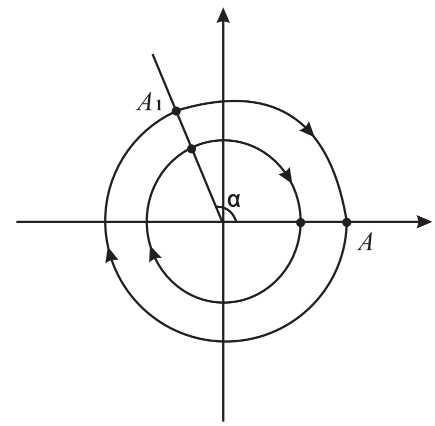

Fig. 1. Periodic orbits of $(1.2)|_{\varepsilon=0}$.

$$M_\alpha(h) = \frac{H_x^-(A, \alpha)}{H_x^+(A, \alpha)} \int_{L_\alpha^+(h)} q^+ dx - p^+ dy + \int_{L_\alpha^-(h)} q^- dx - p^- dy, \ \ h \in J(\alpha). \tag{1.3}$$

Further, if $M_\alpha(h_0) = 0$ and $\frac{\partial M_\alpha}{\partial h}|_{h=h_0} \neq 0$ for some $h_0 \in J(\alpha)$, then for small $\varepsilon > 0$ system (1.2) has a unique limit cycle near $L_\alpha(h_0)$. If h_0 is a zero of $M_\alpha(h)$ having an odd multiplicity, then for small $\varepsilon > 0$ system (1.2) has at least one limit cycle near $L_\alpha(h_0)$. Also, if $M_\alpha(h)$ has at most k zeros counting multiplicity in h on the interval $J(\alpha)$, then system (1.2) has at most k limit cycles bifurcating from the open annulus $\bigcup_{h \in J(\alpha)} L_\alpha(h)$.

We consider the piecewise near-Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 - x + \varepsilon p_n^+(x, y) \\ -b + y + ax + \varepsilon q_n^+(x, y) \end{pmatrix}, & (x, y) \in \Sigma_\alpha^+, \\ \begin{pmatrix} y + \varepsilon p_n^+(x, y) \\ -x + \varepsilon q_n^+(x, y) \end{pmatrix}, & (x, y) \in \Sigma_\alpha^-, \end{cases} \tag{1.4}$$

with the non-regular separation Σ_α defined in (1.1) and Hamiltonian functions

$$H^+(x, y, \alpha) = y - xy + bx - \frac{a}{2}x^2, \ \ H^-(x, y, \alpha) = \frac{1}{2}\big(x^2 + y^2\big), \tag{1.5}$$

where,

$$a = 2\cot\alpha, \ \ b = \frac{\sin\alpha}{1 - \cos\alpha}, \ \ \alpha \in (0, \pi), \tag{1.6}$$

and

$$p_n^\pm(x, y) = \sum_{i+j=0}^{n} a_{ij}^\pm x^i y^j, \ \ q_n^\pm(x, y) = \sum_{i+j=0}^{n} b_{ij}^\pm x^i y^j. \tag{1.7}$$

For each $\alpha \in (0, \pi)$ system $(1.4)|_{\varepsilon=0}$ has a compound homoclinic orbit L_α^* :

$$\big(x(t), y(t)\big) = \begin{cases} \big(1, (b - a)(1 - e^t)\big), & t \in (-\infty, 0), \\ \big(\cos(t), -\sin(t)\big), & t \in [0, 2\pi - \alpha], \\ \big(e^{2\pi-\alpha-t}(\cos(\alpha) - 1) + 1, \big(a - b + \sin(\alpha)\big) \\ \quad e^{2\pi-\alpha-t} + (b - a)\big), \\ t \in (2\pi - \alpha, +\infty) \end{cases}$$

connecting the hyperbolic saddle $S = (1, b - a)$. Let $L_\alpha^1 = L_\alpha^* \cup S$. In view of Lemma 2.1 in Section 2, for system $(1.4)|_{\varepsilon=0}$ there exists a family of periodic orbits between the origin and the compound homoclinic loop L_α^1 as follows