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Nonlinear resonance and devil's staircase in a forced planer system containing a piecewise linear hysteresis

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ABSTRACT

The Duffing equation describes a periodically forced oscillator model with a nonlinear elasticity. In its circuitry, a saturable-iron core often exhibits a hysteresis, however, a few studies about the Duffing equation has discussed the effects of the hysteresis because of difficulties in their mathematical treatment. In this paper, we investigate a forced planer system obtained by replacing a cubic term in the Duffing equation with a hysteresis function. For simplicity, we approximate the hysteresis to a piecewise linear function. Since the solutions are expressed by combinations of some dynamical systems and switching conditions, a finite-state machine is derived from the hybrid system approach, and then bifurcation theory can be applied to it. We topologically classify periodic solutions and compute local and grazing bifurcation sets accurately. In comparison with the Duffing equation, we discuss the effects caused by the hysteresis, such as the devil's staircase in resonant solutions.

1. Introduction

The Duffing equation describes a typical nonlinear non-autonomous system that provides a rich variety of nonlinear phenomena: a horseshoe structure around a saddle fixed point [1], nonlinear resonance with jump phenomena, bifurcations of periodic solutions, chaotic behavior[2] and so on. It has been already well studied from viewpoints varying from mathematical analyses [3] to control engineering [4].

The circuit corresponding to the Duffing equation is achieved by a resistor, a capacitor and a nonlinear inductor with an external driving force [3,5], see Fig. 1. Conventionally, the current of the inductor is approximated by a third-power polynomial of the magnetic flux. However, a practical saturable-core inductor has a hysteresis between the current and the flux. The Duffing equation containing the hysteresis was investigated, e.g., Hayashi [6] suggested that the hysteresis affects structures of bifurcation sets for periodic solutions.

To analyze a nonlinear system rigorously, piecewise linear (PWL) functions are frequently used to approximate these nonlinear characteristics and, conversely, sometimes utilized to create complex behavior [7]. For example, Nishio and Mori [8] showed chaotic phenomena derived from a two-dimensional circuit with a PWL hysteresis and Kimura et al. [9] presented an application of a PWL system. Kousaka et al. [10] developed a general method to solve bifurcation problems of nonlinear systems containing hysteresis.

Nonsmooth systems appear naturally in many practical systems because many physical phenomena present discontinuities: switching in an electrical circuit [11], firing in the neuronal systems [12] or having impacts in mechanics [13]. Their discontinuities can be approximated by the PWL functions.

In this study, we discuss the behavior observed in a forced planer system obtained by replacing a cubic term in the Duffing equation with a hysteresis function. We regard these properties as a simple PWL function in order to apply the hybrid system approaches. In Section 2, we introduce the Duffing equation with its circuitry. We also provide mathematical preliminaries for the hybrid system with hysteresis in this section. We try to describe the hysteresis by defining departure and arrival sets and determine the relationship among them. Then the system provides a finite-state machine (FSM), which is necessary for constructing the hybrid system. In Section 3, we briefly denote topological classifications of periodic solutions and their bifurcations. In Section 4, we show bifurcation diagrams and response curves of periodic solutions in the system by solving the fixed point equation and the characteristic equation simultaneously. We compare the bifurcation structures between the Duffing equation and the proposed hybrid system and

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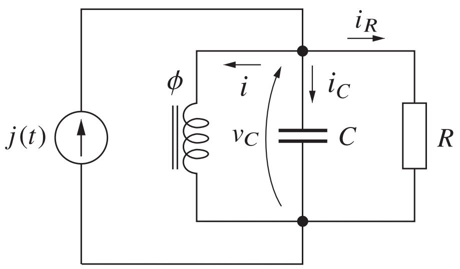

Fig. 1. A forced resonant circuit with a saturable-core inductor.

point out their differences and similarities. We introduce the ratio ρ as a measure to represent a characteristic of the solutions of the proposed system and observe the devil's staircase in the ratio ρ, which cannot be found in the Duffing equation. Finally, we conclude this study in Section 5.

2. Duffing equation containing a piecewise linear hysteresis

2.1. Duffing equation

The resonant circuit shown in Fig. 1 leads to the following equations:

$$
\begin{aligned}
&C\frac{dv_C}{dt} + \frac{v_C}{R} + i = j(t),\\
&N\frac{d\phi}{dt} = v_C,\\
&\quad Ni = G(\phi) = a_1\phi + a_3\phi^3,\\
&\quad j(t) = J_0 + J\cos\omega t,
\end{aligned}
\tag{1}
$$

where ϕ is the magnetic flux of the saturable-core inductor, N is the number of turns of the coil, and G is the characteristic of the inductor that is assumed to be a cubic function. By taking variable transformations such as:

$$
\begin{aligned}
&x = \phi, \tau = \omega t,\\
&k = \frac{1}{\omega RC}, c_1 = \frac{a_1}{N^2\omega^2 C}, c_3 = \frac{a_3}{N^2\omega^2 C},\\
&g(x) = c_1 x + c_3 x^3, B_0 = \frac{J_0}{N\omega^2 C}, \quad B = \frac{J}{N\omega^2 C},
\end{aligned}
$$

we have

$$
\frac{d^2x}{d\tau^2} + k\frac{dx}{d\tau} + g(x) = B_0 + B\cos\tau. \tag{2}
$$

By setting $dx/d\tau$ with y and rewriting τ as t, Eq. (2) can be written as a system of first-order ordinary differential equations on $\boldsymbol{R}^2$:

$$
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -ky - g(x) + B_0 + B\cos t. \tag{3}
$$

We call Eq. (3) the Duffing equation [2]. Several variations of circuit implementation and equations for the Duffing equation have been revisited by Kovacic and Brennan [3].

2.2. Hysteresis of a saturable-core inductor and it's piecewise linear approximation

A saturable-core inductor includes the hysteresis in the relationship between the magnetic flux ϕ and the current i of the inductor, as shown in Fig. 2 (a). In accordance with the magnetic saturation and the remanence of the iron core, a hysteresis loop is formed by varying the flux, i.e., if an increment of the flux exceeds a threshold ϕ_1, a decrement may trace another curve (lower curve in the figure). Also after the negative threshold ϕ_{-1} in the decrement is exceeded, the flux may trace the other curve (upper curve in the figure) in the increment, where the thresholds ϕ_1 and ϕ_{-1} correspond remanences. The appearance of the gap between these curves is the main property of the hysteresis. The state of the saturable-core inductor, magnetized or not, determines which curve the current follows; and therefore, it actually includes a memory in a sense.

We try to approximate the hysteresis to a PWL hysteresis H constructed of two PWL functions shown in Fig. 2(b) [8]. Note $x_{\mathrm{th}}(\pm 1)$ is in accordance with $\phi_{\pm 1}$, and the bending point $x_{\mathrm{bend}}(\pm 1)$ is added. The thickness of the hysteresis loop of the PWL hysteresis is governed by θ.

2.3. Hybrid systems

We adopt the hybrid system approach [14] to consider a dynamical system containing the PWL hysteresis. Although H is not differentiable at $x_{\mathrm{th}}(\pm 1)$ and $x_{\mathrm{bend}}(\pm 1)$ and has the hysteresis loop depending on the state, the approach stated below overcomes these difficulties.

A hybrid system is composed of some smooth dynamical systems and a finite-state machine (FSM). The FSM is a mathematical model used in computation algorithms and this describes the transitions of the finite discrete states (we call *modes*). Each mode gives one dynamical system, and the FSM switches the modes one after another. When we interpret a system as a hybrid system, the total number of the smooth dynamical systems, which is equivalent to the total number of the modes, and the rules of the mode transitions are necessary.

Now we define a hybrid system for the Duffing equation containing the PWL hysteresis. Let m be the total number of the

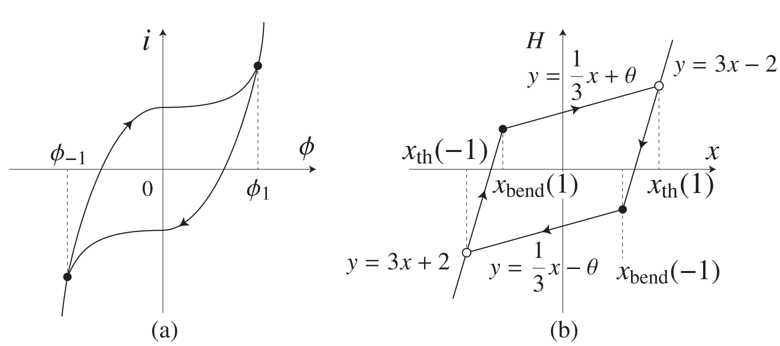

Fig. 2. (a) Schematic illustration of the relationship between magnetic flux and current for the saturable-core inductor, and (b) PWL hysteresis H as an approximated model for (a).