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The threshold of a stochastic delayed SIRS epidemic model with temporary immunity and vaccination

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ABSTRACT

A model of delayed stochastic SIRS type with temporary immunity and vaccination is investigated. The existence and uniqueness of the global positive solution of the model is proved. The threshold of the stochastic SIRS epidemic model is obtained. Compared with the corresponding deterministic model, the threshold affected by the white noise is smaller than the basic reproduction number R_0 of the deterministic system. The vaccination immunity period can also affect the threshold of stochastic and deterministic model. Numerical simulations are carried out to support our theoretical results.

1. Introduction

People have always attached importance to the prevention and the control of the epidemic disease, the study of the epidemic model provides us a powerful tool. The dynamics of diseases by systems of ordinary differential equations without delay is investigated by many researchers. However, inclusion of time delay in models makes them more realistic [1–8]. In epidemiology, vaccination is an important strategy for the elimination of infectious disease. Usually, vaccination immunity period is often considered as a delay factor in constructing epidemic models, and many authors have studied the effects of immunity or vaccination [9–20, 31].

Assuming the population input is a constant and the incidence is of bilinear form, only the susceptible is vaccinated to and only this vaccination is effective, the recovered and vaccinated have immunization period, Li [13] established an SIRS epidemic model with vaccination. The model is expressed by the following ordinary differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) - pS(t) - \beta S(t)I(t) + \gamma I(t-\tau_1)e^{-\mu\tau_1} \\ \quad + pS(t-\tau_2)e^{-\mu\tau_2}, \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\alpha+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) + pS(t) - \mu R(t) - \gamma I(t-\tau_1)e^{-\mu\tau_1} \\ \quad - pS(t-\tau_2)e^{-\mu\tau_2}. \end{cases} \tag{1.1}$$

where $S(t)$ is the number of the individuals susceptible to the disease, $I(t)$ is the infected member and $R(t)$ denotes the members who have recovered from the infection and the members who have been vaccinated. In the model, the parameters have the following biological meanings: A denotes a constant input of new members into the population; μ is the natural death rate; p is the vaccination rate of the susceptible; β is the transmission coefficient between compartments S and I; α is the death rate due to illness; γ is the rate of recovery from infection; τ_1 and τ_2 represent respectively the length of immunity period of the recovered and the vaccinated. A, μ, p, β, α, γ are all positive constants. For this model, Li obtained the basic reproduction number $R_0 = \frac{\beta A}{(\mu+\alpha+\gamma)(\mu+p(1-e^{-\mu\tau_2}))}$ and proved that the dynamics of the model is completely determined by R_0: if $R_0 \le 1$, then model (1.1) has a unique disease-free equilibrium P_0 which is globally stable in R^2_+; if $R_0 > 1$, then P_0 is unstable and model (1.1) has an en-

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demic equilibrium $p_e = (S_e, I_e) = (\frac{\mu+\alpha+\gamma}{\beta}, \frac{A}{\mu+\alpha+\gamma}(1-\frac{1}{R_0}))$, which is globally asymptotically stable.

However, in the natural world, epidemic models are always affected by the environment noise. Thus, it is necessary to reveal how the environmental noise affects the epidemic model [21–25]. Based on model (1.1) and assuming that the stochastic perturbation is proportional to each variable value, we get the following stochastic SIRS model driven by white noise:

$$\begin{cases} dS(t) = [A - \mu S(t) - pS(t) - \beta S(t)I(t) + \gamma I(t-\tau_1)e^{-\mu\tau_1} \\ \quad + pS(t-\tau_2)e^{-\mu\tau_2}]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\alpha+\gamma)I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dR(t) = [\gamma I(t) + pS(t) - \mu R(t) - \gamma I(t-\tau_1)e^{-\mu\tau_1} \\ \quad - pS(t-\tau_2)e^{-\mu\tau_2}]dt + \sigma_3 R(t)dB_3(t). \end{cases} \tag{1.2}$$

where $B_i(t)$ (i=1,2,3)are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the filtration $\mathscr{F}_t \geq 0$, satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets).

Notice that the first two equations in system (1.2) do not depend on the third equation and so this equation can be omitted without loss of generality. Thus, we here only discuss the following model:

$$\begin{cases} dS(t) = [A - \mu S(t) - pS(t) - \beta S(t)I(t) + \gamma I(t-\tau_1)e^{-\mu\tau_1} \\ \quad + pS(t-\tau_2)e^{-\mu\tau_2}]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\alpha+\gamma)I(t)]dt + \sigma_2 I(t)dB_2(t), \end{cases} \tag{1.3}$$

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of the positive solution of model (1.3). The sufficient condition for the extinction of the disease is obtained in Section 3. In Section 4, we establish the sufficient condition for persistence of the disease. In Section 5, we make simulations to confirm our results.

2. Existence and uniqueness of the positive solution

In this section, we show that model (1.3) has a unique global positive solution by the Khasminskii–Mao theorem [26,27]. The solution of model (1.3) may explode in a finite time, since the coefficients don't satisfy the linear growth condition, even though they satisfy the locally Lipschitz condition. Khasminskii and Mao gave the Lyapunov function argument, which is a powerful test for non-explosion of solutions without the linear growth condition and referred to as the Khasminskii–Mao theorem. We will give the proof in the following.

Theorem 2.1. *For any given initial value $S(\xi_1) \geq 0$ and $I(\xi_2) \geq 0$ for all $\xi_1 \in [-\tau_1, 0)$, $\xi_2 \in [-\tau_2, 0)$ with $S(0) > 0$, $I(0) > 0$, system (1.3) has a unique global solution $(S(t), I(t)) \in \mathbb{R}^2_+$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of system (1.3) are Lipschitz continuous, then for any given initial value $S(\xi_1) \geq 0$ and $I(\xi_2) \geq 0$ for all $\xi_1 \in [-\tau_1, 0)$, $\xi_2 \in [-\tau_2, 0)$ with $S(0) > 0$, $I(0) > 0$, there is a unique local solution $(S(t), I(t))$ on $t \in [-\tau, \tau_e)$, where $\tau = \max(\tau_1, \tau_2)$ and τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ *a.s.* Let $k_0 > 1$ be sufficiently large so that $S(0)$, $x(0) \in [\frac{1}{k0}, k_0]$. For each integer $k \geq k_0$, define the stopping time as

$$\tau_k = \inf\left\{t \in [0, \tau_e) : S(t) \notin \left(\frac{1}{k}, k\right) \ \ or\ x(t) \notin \left(\frac{1}{k}, k\right)\right\}.$$

Throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ *a.s.* Thus to show $\tau_e = \infty$ *a.s*, we need only to show $\tau_\infty = \infty$ *a.s.*

If $\tau_\infty \neq \infty$, then there is a pair of constants $T > 0$ and $\delta \in (0, 1)$ such that

$$P(\tau_\infty \leq T) > \delta.$$

Hence there is an integer $k_1 \geq k_0$ such that for all $k \geq k_1$,

$$P(\tau_k \leq T) \geq \delta. \tag{2.1}$$

Define a C^2−function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as

$$V(S, I) = \left(S - a - a\ln\frac{S}{a}\right) + (I - 1 - \ln I) + \gamma e^{-\mu\tau_1}\int_{t-\tau_1}^{t} I(s)ds + pe^{-\mu\tau_2}\int_{t-\tau_2}^{t} S(s)ds,$$

where a is a positive constant to be determined later. The nonnegativity of this function can be seen from $u - 1 - lnu$ for all $u > 0$. Using Itô formula, we obtain

$$dV(S, I) = LV(S, I)dt + \sigma_1(S-a)dB_1(t) + \sigma_2(I-1)dB_2(t),$$

where

$$\begin{aligned} LV(S, I) = {} & \left(1 - \frac{a}{S(t)}\right)(A - \mu S(t) - pS(t) - \beta S(t)I(t) \\ & + \gamma I(t-\tau_1)e^{-\mu\tau_1} + pS(t-\tau_2)e^{-\mu\tau_2}) \\ & + \left(1 - \frac{1}{I}\right)(\beta S(t)I(t) - (\mu+\alpha+\gamma)I(t)) + \frac{1}{2}a\sigma_1^2 \\ & + \frac{1}{2}\sigma_2^2 + \gamma e^{-\mu\tau_1}I(t) - \gamma e^{-\mu\tau_1}I(t-\tau_1) + pe^{-\mu\tau_2}S(t) \\ & - pe^{-\mu\tau_2}S(t-\tau_2). \\ \leq {} & A - \mu S(t) - pS(t) - A\frac{a}{S(t)} + \mu a + pa + a\beta I(t) \\ & - (\mu+\alpha+\gamma)I(t) + (\mu+\alpha+\gamma) \\ & - \beta S(t) + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \gamma e^{-\mu\tau_1}I(t) + pe^{-\mu\tau_2}S(t) \\ = {} & \left(A + \mu a + pa + \mu + \alpha + \gamma + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2\right) - \mu S(t) \\ & - A\frac{a}{S(t)} - \beta S(t) - p(1 - e^{-\mu\tau_2}S(t)) \\ & + (a\beta - \mu - \alpha - \gamma(1 - e^{-\mu\tau_1}))I(t). \end{aligned}$$

Choose $a = \frac{\mu+\alpha+\gamma(1-e^{-\mu\tau_1})}{\beta}$, such that $[a\beta - \mu - \alpha - \gamma(1 - e^{-\mu\tau_1})]I(t) = 0$, then

$$LV(S, I) \leq A + \mu a + pa + \mu + \alpha + \gamma + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2 =: K.$$

Therefore,

$$dV(S, I) \leq Kdt + \sigma_1(S-a)dB_1(t) + \sigma_2(I-1)dB_2(t), \tag{2.2}$$

Integrating both sides of (2.2) from 0 to $\tau_k \wedge T$ and taking expectations, obtain

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T)) \leq V(S(0), I(0)) + KE(\tau_k \wedge T),$$

Therefore,

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T)) \leq V(S(0), I(0)) + KT. \tag{2.3}$$

set $\Omega_k = \{\tau_k \leq T\}(k \geq k_1)$ and by (2.1) $P(\Omega_k) \geq \delta$. Note that for every $\omega \in \Omega_k$, there is at least one of $S(\tau_k, \omega)$ and $x(\tau_k, \omega)$ equals either k or $\frac{1}{k}$, then

$$V\big(S(\tau_k, \omega), x(\tau_k, \omega)\big) \geq \left(k - a - a\ln\frac{k}{a}\right) \wedge \left(\frac{1}{k} - a - a\ln\frac{1}{ka}\right) \wedge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right).$$