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On the rational limit cycles of Abel equations

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ABSTRACT

In this paper, we deal with Abel equations: $\frac{dx}{dy} = A(x)y^2 + B(x)y^3$, where $A(x)$ and $B(x)$ are real polynomials. If a solution $y = \varphi(x)$ of the above equations satisfies that $\varphi(0) = \varphi(1)$, then we say that it is a periodic solution. If a periodic solution is isolated, then we call it a limit cycle. If a limit cycle $y = \varphi(x)$ is a rational function but not a polynomial, then we call it a nontrivial rational limit cycle.

Firstly, we study the existence of nontrivial rational limit cycles. We prove that there exist Abel equations, which have at least two nontrivial rational limit cycles, and there also exists other Abel equations, which have at least one nontrivial rational limit cycle and one non-rational limit cycle. Secondly, we discuss the relation between the existence of nontrivial rational limit cycle and the degrees of $A(x)$ and $B(x)$. Finally we show that the multiplicity of a nontrivial rational limit cycle can be unbounded.

1. Introduction and the main results

In this paper, we consider Abel equations,

$$\frac{dy}{dx} = A(x)y^2 + B(x)y^3 \tag{1}$$

where x and y are real variables, $A(x)$ and $B(x)$ are real polynomials. These equations are interesting since they occur in many models of real phenomena (see for instants [2,9,12]), or are a tool to study several subcases of Hilbert 16th problem on the number of limit cycles of planar polynomial differential equations (see [7,15]).

Denote by $y = \psi(x, y_0)$ the solution of system (1) such that $\psi(0, y_0) = y_0$. We say that the solution $y = \psi(x, y_0)$ of system (1) is periodic if $\psi(1, y_0) = y_0$. If $\psi(x, y_0)$ is well defined in [0, 1], then we can define a map $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\Psi(y_0) = \psi(1, y_0) - y_0$. Obviously, $\Psi(y_0) = 0$ if and only if system (1) has a periodic solution starting at y_0. The periodic solution $\psi(x, y_0)$ is called as a limit cycle if y_0 is an isolated zero of $\Psi(y_0)$, at the same time, the multiplicity of y_0 as a zero of $\Psi(y_0)$ is called as the multiplicity of the associated limit cycle. Specially, when the multiplicity of a limit cycle is 1, this limit cycle will be said to be simple or hyperbolic.

The problem of the number of limit cycles of system (1) has been widely studied (see for instance [14] or [10] and therein). Of course, there are authors who consider other related problems. For example, in [3–6,17], the authors consider the problem of centers, that is, near the trivial solution $y = 0$, all the solutions of system (1) are periodic; in [1,8,13], the authors consider the problems of centers or limit cycles under the conditions that the functions $f(x)$ and $g(x)$ are trigonometric polynomials or only analytic functions.

Specially, in [11], J. Giné, M. Grau and J. Libre consider the problem of polynomial limit cycles of the following system

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + \cdots + A_n(x)y^n,$$

where $A_i(x)$ for $i = 0, 1, 2 \ldots, n$ are real polynomials in x. Of course here a limit cycle $y = \psi(x, y_0)$ is called as a polynomial limit cycle if the function $\psi(x, y_0)$ is a polynomial in x,

When $n = 3$ and $A_0(x) = A_1(x) = 0$, the above system becomes system (1). Now for system (1), the results in [11] can be written as

Theorem 1. *For system (1), the following statements hold.*

(a) *Any polynomial limit cycle of system (1) has the form $y = c$. Furthermore, if there exists a polynomial limit cycle $y = c$, $c \neq 0$, then except for the polynomial solution $y = 0$, there is no other polynomial solution, thus no other polynomial limit cycle.*
(b) *For any given integer k with $k \geq 2$, there exist polynomials $A(x)$ and $B(x)$ such that $y = 0$ is a polynomial limit cycle of multiplicity k of system (1).*

It is worth pointing out that if a polynomial limit cycle $y = c$, $c \neq 0$, exists, then it is hyperbolic. The proof will be found in the next section.

Note that in [11], the authors only consider polynomial solutions, but what happens when we consider other solutions, such as algebraic function solutions? Recall the a function $y = \phi(x)$ is

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called algebraic if and only if there exists a polynomial $f(x, y)$ such that $f(x, \phi(x)) \equiv 0$. We call that a limit cycle $y = \psi(x, y_0)$ of system (1) is an algebraic limit cycle if $\psi(x, y_0)$ is algebraic.

To solve the problem of algebraic limit cycles, it is natural for us to start from the simple case: rational case, in other words, the limit cycle $y = \psi(x, y_0)$ can be written as $y = \frac{q(x)}{p(x)}$, where $p(x)$, $q(x)$ are polynomials and $(p(x), q(x)) = 1$. Since we are only interested in rational limit cycles, which are not polynomial limit cycles, we will call polynomial limit cycles trivial rational limit cycles and other rational limit cycles nontrivial rational limit cycles. In this paper we will show that new phenomena occur.

As usual, we consider the problem of rational limit cycles from three directions: the existence and non-existence of rational limit cycles, the number of rational limit cycles and the multiplicity of rational limit cycles.

The rest of this paper is structured as follows. In Section 2, we give some preliminaries, some of which concern whether a rational function is a rational limit cycle and some of which are useful tools to prove our theorems. In Section 3, we consider the existence of nontrivial rational limit cycles and the degrees of $A(x)$ and $B(x)$ in system (1), and the existence of two nontrivial rational limit cycles. In Section 4, we prove that he multiplicity of nontrivial rational limit cycle is unbounded. All the results in Section 3 and 4 are different from that in the polynomial case.

2. Preliminaries

The following lemma gives the sufficient and necessary conditions that a rational function can be a periodic solution of system (1).

Lemma 2. *The nonzero rational function $y = \frac{q(x)}{p(x)}$ is a periodic solution of system (1) if and only if all the following three conditions hold:*

(a) $q(x) = c_1$, *where c_1 is a nonzero constant;*
(b) $c_1 B(x) + \frac{p(x)p'(x)}{c_1} + p(x)A(x) = 0$;
(c) $p(0) = p(1)$ *and $p(x)$ has no zero in* $[0, 1]$.

Proof. If $y = \frac{q(x)}{p(x)}$ is a periodic solution of system (1), then obviously $p(x)$ has no zero in [0, 1]. Denote by $F(x, y) = p(x)y - q(x)$, then along the curve $\{F(x, y) = 0\}$, the derivative of $F(x, y)$ in x is zero, in other words,

$$\frac{dF}{dx} = p'(x)y - q'(x) + p(x)(A(x)y^2 + B(x)y^3) = 0.$$

Notice that $F(x, y)$ is irreducible, so there exists a polynomial $k(x, y)$ so that

$$p'(x)y - q'(x) + p(x)(A(x)y^2 + B(x)y^3) = k(x, y)F(x, y). \tag{2}$$

The left-hand side of formula (2) is a polynomial of degree 3 in y and $F(x, y)$ is a polynomial of degree 1 in y, so we can suppose that $k(x, y) = k_0(x) + k_1(x)y + k_2(x)y^2$, where k_0, k_1 and k_2 are polynomials in x. Via Comparing the coefficients of y^k, $k = 0, 1, 2, 3,$ in formula (2), we obtain

(i) $-q'(x) = -k_0(x)q(x)$;
(ii) $p'(x) = k_0(x)p(x) - k_1(x)q(x)$;
(iii) $p(x)A(x) = k_1(x)p(x) - k_2(x)q(x)$;
(iv) $p(x)B(x) = k_2(x)p(x)$.

From (i), $q(x)$ is a factor of $q'(x)$, that is, $q(x)|q'(x)$. This fact implies that $q(x)$ is a constant and we denote it by $q(x) = c_1 \in \mathbb{R}$. If $c_1 = 0$, then $y = \frac{q(x)}{p(x)} = 0$. This is impossible, so we have that $c_1 \neq 0$, the condition (a) holds. Furthermore, $y = \frac{q(x)}{p(x)} = \frac{c_1}{p(x)}$ is a periodic solution, thus $p(0) = p(1)$, the condition (c) holds.

From (ii), $k_1(x) = -\frac{p'(x)}{c_1}$; from (iv), $k_2(x) = B(x)$. Substituting them to (iii), we have

$$c_1 B(x) + \frac{p(x)(p'(x)}{c_1} + p(x)A(x) = 0.$$

The condition (b) holds.

On the contrary, if all the three conditions (a),(b),(c) hold, then one can easily check that the rational function $y = \frac{c_1}{p(x)}$ is a periodic solution of system (1). □

Since $q(x) = c_1$ is a nonzero constant, without loss of generality, in the rest of this paper, we will suppose that $q(x) = 1$. To decide a periodic solution of system (1) is a hyperbolic limit cycle, we need the following lemma, whose proof can be find in [11] (Lemma 5).

Lemma 3. *Consider the differential equation $\frac{dy}{dx} = \mathcal{F}(x, y)$, where $\mathcal{F}(x, y)$ is a function of class $\mathcal{C}^2$ in $\mathbb{R}^2$. Assume that $y = \varphi(x)$ is a periodic orbit of this equation, then it is a hyperbolic limit cycle if and only if $\mathcal{D}_1(1) \neq 0$, where*

$$\mathcal{D}_1(x) = \int_0^x \frac{\partial \mathcal{F}}{\partial y}(t, \varphi(t))\, dt.$$

We firstly use Lemma 3 to deal with polynomial limit cycles:

Corollary 1. *If $y = c$, $c \neq 0$, is a limit cycle of system (1), then it must be a hyperbolic limit cycle.*

Proof. Since $y = c$ is a solution of system (1), by Lemma 2, we have that $A(x) = -cB(x)$ and system (1) becomes

$$\frac{dy}{dx} = B(x)y^2(y - c).$$

If $\int_0^1 B(x)dx \neq 0$, then

$$\mathcal{D}_1(1) = \int_0^1 \frac{\partial B(x)y^2(y - c)}{\partial y}(t, c)\, dt = c^2 \int_0^1 B(t)dt \neq 0.$$

By Lemma 3, the periodic solution $y = c$ is a hyperbolic limit cycle.

If $\int_0^1 B(x)dx = 0$, then $\mathcal{D}_1(1) = 0$, but now $A(x)$ and $B(x)$ satisfy the composition condition (see for example [3]), all the solutions are periodic, so $y = c$ is not a limit cycle. This case does not occur. □

Secondly we use Lemma 3 to deal with nontrivial rational limit cycles:

Corollary 2. *If $y = \frac{1}{p(x)}$ is a periodic solution of system (1), then it is a hyperbolic limit cycle if and only if $\int_0^1 \frac{B(x)}{p^2(x)}dx \neq 0$.*

Proof. By direct calculations, we have

$$\mathcal{D}_1(1) = \int_0^1 \frac{\partial A(x)y^2 + B(x)y^3)}{\partial y}\left(t, \frac{1}{p(t)}\right) dt = \int_0^1 \frac{2A(x)p(x) + 3B(x)}{p^2(x)}\, dx.$$

By Lemma 2, $p(0) = p(1)$ and $p(x)A(x) = -B(x) - p(x)p'(x)$, so

$$\mathcal{D}_1(1) = \int_0^1 \frac{B(x)}{p^2(x)} - 2\frac{p'(x)}{p(x)}\, dx = \int_0^1 \frac{B(x)}{p^2(x)}\, dx,$$

which finishes the proof. □

The following well-known result of linear algebra will be used in Section 4.

Theorem 4. *Let $v_1, v_2, \cdots, v_n$ be elements of a vectorial space E endowed with an inner product $\langle,\rangle$. Then*