Chaos, Solitons and Fractals 110 (2018) 55–63

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Co-existence of in-phase oscillations and oscillation death in environmentally coupled limit cycle oscillators

Umesh Kumar Verma[a], Neeraj Kumar Kamal[a,b], Manish Dev Shrimali[a,*]

[a] Department of Physics, Central University of Rajasthan, Ajmer, 305 817, India
[b] Anugrah Memorial College, Gaya, 823 001, India

ARTICLE INFO

Article history:
Received 26 September 2017
Revised 4 March 2018
Accepted 4 March 2018

Keywords:
Amplitude death
Oscillation death
Bistability
Environmental coupling

ABSTRACT

We study the emergent dynamics of limit cycle oscillators coupled indirectly via a dynamic environment. We report the co–existence of in–phase oscillations and oscillation death in the parameter plane, which is observed for the first time in indirectly coupled systems. It is found that the emergent dynamics of this system crucially depend on both the decay parameter of the environment and the density of the mean–field coupling. Also, we found a distinct route to suppressed oscillation state, namely, amplitude death and oscillation death, from the oscillatory solutions. Our numerical results are consistent with the analytical results obtained from linear stability analysis and further corroborated by the experimental results obtained from the electronic circuit implementation of the system.

1. Introduction

Models of coupled oscillators are not only the prototype of complex systems but also has been used to gain more understanding of these systems. The two most prominent emergent behavior of coupled oscillators are, *synchronization* [1] and *oscillation suppression* [2,3]. Synchronization is the adjustment of a rhythm of the oscillators due to coupling and has been a widely studied topic [1]. Oscillation suppression is classified into two classes, namely, amplitude death(AD) and oscillation death(OD), due to their manifestation and origin. AD is the stabilization of trivial steady state of uncoupled oscillators due to coupling [2], however, in OD the emergent steady state break the inherent symmetry present in the oscillators due to the coupling term [3]. From an application point of view, AD has potential applications in the cases where suppression of unwanted oscillation is necessary, e.g., laser [4], oceanography [5], and neuronal systems [6], whereas OD has applications in an understanding of biological system including neural networks [7], genetic oscillators [8] and cell differentiation [9]. These two are structurally different phenomena but can simultaneously occur in a coupled system. Over that past few years, several works have been devoted to study the transition from AD to OD in coupled oscillators for different coupling scheme, e.g. simple diffusive coupling [3], mean-field diffusive coupling [10–13], time delay coupling [15], conjugate coupling [14], direct and indirect coupling [16,17], repulsive coupling [18] and indirect coupling [19–21].

It has been found that in many physical, chemical and biological systems, the components interact with each other indirectly through common environment and the direct interactions among the components are almost absent. For instance, the phenomenon of quorum sensing or cell-cell communication realizes when bacterial colony releases chemical signal molecules (called auto-inducers) into the environment. The interaction of cell and environment may occur through the diffusion and transport of chemical signal molecules across the cell membrane, which allow the bacteria to sense a critical cell mass and in response, to the activation of receptors on the cell membrane [22]. In the case of chemical oscillations, the catalyst loaded reactants in a medium exchange chemicals with the surrounding medium [23]. Similarly, an ensemble of cold atoms interacts through a coherent electromagnetic field [24] and in case of neuronal oscillator indirect communication exist where concentrations of neurotransmitters released by each cell stimulate collective rhythms in a population of circadian oscillation [25]. There are various studies which have been focused on the collective dynamics of these systems where the exchange of information among the systems is indirect [26–38].

The coexistence of oscillatory solutions along with steady state solutions has been found in literature and also has a great significance in biological oscillators [39–41]. This type of bistability has also been observed in a theoretical model of chemical oscillators [42,43]. Also, in many physical systems, this coexistence of

* Corresponding author.
E-mail address: shrimali@curaj.ac.in (M.D. Shrimali).

https://doi.org/10.1016/j.chaos.2018.03.009
0960-0779/

in-phase oscillatory solutions and OD state has been found [44–47]. Such coexisting steady state and in–phase oscillation found in chemical oscillators, laser systems as well as chaotic systems such as Lorenz, Rössler and Pikovsky-Rabinovich is due to the symmetry breaking and symmetry preserving nature of the coupling respectively. In this context, the coexistence of these two solutions is not observed in an indirectly coupled system of oscillators where direct interaction is absent [19]. Given the importance of indirect coupling, it is imperative to design a scheme of coupling where there is a possibility of an existence of this bistable state.

In this work, we introduce a coupling scheme which is not only indirect but also allows the coexistence of in-phase oscillatory solutions and OD in a system of limit cycle oscillators. We have studied the emergent states of Stuart-Landau oscillators and Van-der Pol oscillators for this coupling scheme. The outline of this paper is as follows: In Section 2, we introduce our model of indirectly coupled oscillators and then discuss numerical as well as analytical results obtained from linear stability analysis, for both Stuart-Landau and Van-der Pol oscillators. Next, in Section 3 we present the electronic circuit implementation of our model for Van-der Pol oscillators and demonstrate the different dynamical state of the indirectly coupled systems through experimental time series plot. Finally, in Section 4, we will discuss the results and present the conclusions.

2. Model of two indirectly coupled limit cycle oscillators

We consider N identical two–dimensional limit cycle oscillators which are indirectly coupled via a dynamic agent in a local medium/environment. The dynamics of this system is given by,

$$\begin{aligned} \dot{x}_i &= f_1(x_i, y_i) + \epsilon(s - x_i), \\ \dot{y}_i &= f_2(x_i, y_i), \\ \dot{s} &= -\gamma s + \epsilon(Q\bar{x} - s), \end{aligned} \tag{1}$$

where, $i(=1,\dots,N)$ is the oscillator index. $(x_i,\ y_i)$ are the state variables of the ith limit cycle oscillators whose time evolution is specified by the functions $f_1(x_i,\ y_i)$ and $f_2(x_i,\ y_i)$ respectively. 's' is the state variable of the common environment of these oscillators. The dynamics of the environment is modeled by an over damped oscillator with damping co–efficient, γ. To remain active, state variable, s of the environment interacts with the x component of the limit cycle oscillators. ϵ is the coupling strength of diffusion for both, the oscillators and the local medium. $\bar{x}(=\frac{1}{N}\sum_{i=1}^{N}x_i)$ is the mean–field of the N oscillators and Q is the intensity of mean–field interaction influencing the dynamics of the local medium. In this coupling scheme, interaction among limit cycle oscillators is through the dynamic agent s of the medium, which gets feedback from the diffusion of some intensity of the mean-field of state variable x. This makes the coupling or exchange of information among the oscillators indirect.

Moreover, the state variable of oscillators, x and state variable of environment, s, represents common particle species that can freely diffuse in the system and allow individual oscillators to communicate with each other [37,38]. The specific realizations of x and s depend on the context. In the BZ reaction, it represents chemical species that diffuse between auto-catalytic beads and similarly in metabolic oscillations, it represents common metabolites that diffuse between cells [38]. The analogy of this coupling scheme can also be found in [39], where the molecules released from the cells are diffused in the local medium constitutes a mean–field which in turn effect the collective dynamics of the cells through indirect interactions.

2.1. Stuart–Landau oscillators

We first consider N identical indirectly coupled Stuart–Landau (SL) oscillators. The dynamics of the system for this coupling scheme can be written as,

$$\begin{aligned} \dot{x}_i &= (1 - x_i^2 - y_i^2)x_i - \omega y_i + \epsilon(s - x_i), \\ \dot{y}_i &= (1 - x_i^2 - y_i^2)y_i + \omega x_i \\ \dot{s} &= -\gamma s + \epsilon(Q\bar{x} - s), \end{aligned} \tag{2}$$

where, $i = 1, \dots, N$ is oscillator index. The frequency of the oscillators is $\omega = 2$.

We first consider the case for which $N = 2$. From Eq. (2), it is clear that for $N = 2$, system has a trivial fixed point, which is the origin (0,0,0,0,0), and two additional coupling dependent nontrivial fixed points: (i) Inhomogeneous steady state (IHSS) $(x^\diamond, y^\diamond, -x^\diamond, -y^\diamond, s^\diamond)$ where, $x^\diamond = y^\diamond\left(\frac{-\epsilon\pm\sqrt{\epsilon^2-4\omega^2}}{2\omega}\right)$, $y^\diamond = \sqrt{\frac{\epsilon-2\omega^2\pm\sqrt{\epsilon^2-4\omega^2}}{2\epsilon}}$ and $s^\diamond = 0$ and (ii) Homogeneous steady states (HSS); $(x^\dagger,\ y^\dagger,\ x^\dagger,\ y^\dagger,\ s^\dagger)$, where, $x^\dagger = \left(\frac{-\epsilon'\pm\sqrt{\epsilon'^2-4\omega^2}}{2\omega}\right)y^\dagger$, $y^\dagger = \sqrt{\frac{(\epsilon'-2\omega^2)\pm\sqrt{\epsilon'^2-4\omega^2}}{2\epsilon'}}$, $s^\dagger = \frac{Q\epsilon}{\gamma+\epsilon}x^\dagger$ and $\epsilon' = 1 - \frac{Q\epsilon}{\epsilon+\gamma}$. The stabilization of trivial fixed point (origin) lead to the AD state in the system, while stabilization of nontrivial fixed points lead to the OD state.

To understand the different emergent dynamical states of the system theoretically, we first analyze the stability of the fixed points through linear stability analysis. For that, we calculate the eigenvalues of 5×5 Jacobian matrix, calculated at the fixed points. The characteristic equation of the system at the trivial fixed point (0, 0, 0, 0, 0), is given by,

$$(\lambda^2 + c_1\lambda + c_0)(\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0) = 0, \tag{3}$$

where, $c_1 = \epsilon - 2, c_0 = 1 + \omega^2 - \epsilon, d_2 = \gamma + 2\epsilon - 2, d_1 = (1 - Q)\epsilon^2 + (\gamma - 3)\epsilon + (\omega^2 - 2\gamma - 1)$ and $d_0 = (Q - 1)\epsilon^2 + (\omega^2 - \gamma + 1)\epsilon + \gamma(1 + \omega^2)$.

The stability of origin can be computed from the coefficients of the above characteristic equation by the method given in ref. [48]. The exact locus of Hopf bifurcation points HB1 and HB2, through which origin get stabilized, in the parameter plane is obtained by putting $c_1 = 0$ and $d_1d_2 - d_0 = 0$ respectively,

$$\begin{aligned} \epsilon_{HB1} &= 2, \\ \gamma_{HB2} &= \frac{8\epsilon - 4 - \epsilon^2(3 - Q) + \sqrt{m}}{2(\epsilon - 2)}, \end{aligned} \tag{4}$$

with $m = (1 + 2Q + Q^2)\epsilon^4 - (1 + Q)4\epsilon^3 + 4\epsilon^2 - 4\omega^2(\epsilon - 2)^2$. The pitchfork bifurcation points PB1 and PB2, can also be obtained from Eq. (3), by putting $c_0 = 0$ and $d_0 = 0$ respectively,

$$\begin{aligned} \epsilon_{PB1} &= 1 + \omega^2 = \epsilon^*, \\ \epsilon_{PB2} &= \frac{(\gamma - \epsilon^*) - \sqrt{(\epsilon^* - \gamma)^2 - 4\gamma\epsilon^*(Q - 1)}}{2(Q - 1)}. \end{aligned} \tag{5}$$

The critical value of γ for which the origin stabilizes through Hopf bifurcation is γ_{HB2}. Also, ϵ_{PB1} and ϵ_{PB2} are the critical value of ϵ at which pitchfork bifurcation occur from the origin. These two pitchfork bifurcations give rise to IHSS and HSS solutions respectively. The stability of these states can be computed from the eigenvalues of the Jacobian matrices. The characteristic equation corresponding to the IHSS, $(x^\diamond, y^\diamond, -x^\diamond, -y^\diamond, s^\diamond)$ is,

$$(\lambda^2 + c_1^\diamond\lambda + c_0^\diamond)(\lambda^3 + d_2^\diamond\lambda^2 + d_1^\diamond\lambda + d_0^\diamond) = 0, \tag{6}$$

where, $c_1^\diamond = \epsilon + 4\beta - 2$, $c_0^\diamond = 1 + \omega^2 - 4\beta + 3\beta^2 + \epsilon(x^{\diamond2} + 3y^{\diamond2} - 1)$, $d_2^\diamond = 4\beta + \gamma + 2(\epsilon - 1)$, $d_1^\diamond = 1 - 4\beta + 3\beta^2 + 4\gamma\beta + \epsilon(5x^{\diamond2} + 7y^{\diamond2}) + p_1$, $d_0^\diamond = 3\gamma\beta^2 - 4\gamma\beta - 4\epsilon\beta + 3\epsilon\beta^2 + \gamma\epsilon(x^{\diamond2} + 3y^{\diamond2}) + \epsilon^2(x^{\diamond2} + 3y^{\diamond2} - Qx^{\diamond2} - 3Qy^{\diamond2}) + p_0$, $\beta = x^{\diamond2} + y^{\diamond2}$, $p_1 = \epsilon(\gamma - 3) + \epsilon^2(1 - Q) + \omega^2 - 2\gamma$, $p_0 = \gamma + \epsilon(1 - \gamma) + \epsilon^2(Q - 1) + \omega^2(\gamma + \epsilon)$.