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Noise-induced bursting and chaos in the two-dimensional Rulkov model

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ABSTRACT

We study an effect of random disturbances on the discrete two-dimensional Rulkov neuron model. We show that close to the Neimark–Sacker bifurcation, the increasing noise can cause the transition from the noisy quiescence with small-amplitude oscillations near the stable equilibria to the stochastic bursting with large-amplitude spikes. Mean values and variations of the interspike intervals are studied in dependence of the noise intensity. To study the noise-induced bursting, the analytical approach based on the stochastic sensitivity functions technique and confidence ellipses method is applied. On the basis of the largest Lyapunov exponents, we show how the noise-induced transition from the quiescence to stochastic bursting regime is accompanied by the transformation of dynamics from regular to chaotic.

1. Introduction

In the actively developing field of research related to biological modeling, a special role is played by the study of dynamic models of the neuronal activity. These models reflect a wide diversity of the regimes of the neuronal activity, and therefore possess complex excitable attractors and exhibit unexpected nonlinear phenomena [1–3]. The continuous-time models, such as Fitzhugh–Nagumo [4–8], Hodgkin–Huxley [9–13], Morris–Lecar [14–17] and Hindmarsh–Rose [18–23] models, were extensively studied by many authors in both deterministic and stochastic cases. While the theory of continuous-time neuron models using differential equations has been widely developed, less attention has been devoted to the study of map-based discrete-time models [24].

The discrete-time Rulkov system [25] was one of the first phenomenological models which demonstrate basic types of the neural activity, such as the quiescence, the tonic spiking and bursting. Mathematically, the two-dimensional Rulkov model exhibits various bifurcations and attractors [26,27]. This model is actively used in the study of the dynamics of neural networks [28–31]. Even in the one-dimensional case, the Rulkov model exhibits interesting phenomena under the influence of random disturbances [32].

The aim of the present paper is to study how the two-dimensional Rulkov model responds to random perturbations. Our analysis is focused on the parametric zone near the Neimark–Sacker bifurcation.

An analysis of the stochastic phenomena in the map-based dynamical systems attracts attention of many researchers (see, e.g.[33–36]). However, until now the main research method is the direct numerical simulation of the random trajectories that is time-consuming in the parametric study. A rigorous theoretical description of the dynamics of probabilistic distributions of solutions of the stochastic discrete systems is given by the Perron–Frobenius equation [36]. An analytical solution of such functional equations is available only in the exceptional cases. A constructive method of the approximation of the probabilistic distributions based on the stochastic sensitivity functions (SSF) technique has been proposed in [37]. In the present paper, we apply this technique to the analysis of the noise-induced bursting in the two-dimensional Rulkov model. A short overview of the stochastic sensitivity function technique and method of the confidence ellipses is given in the Appendix.

The present paper is organized as follows.

In Section 2, we give a short summary of dynamical regimes for the deterministic 2D Rulkov model in the zone of the Neimark–Sacker bifurcation connected with the loss of stability of the equilibrium and birth of the stable invariant curve. In the zone of stable equilibria, sub- and superthreshold regimes are discussed. In the zone of the stable closed invariant curves, a phenomenon of the Canard explosion is illustrated.

Section 3 is devoted to the study of the noise-induced generation of the stochastic bursting in the zone where the initial deterministic model has the stable equilibrium as a single attractor. We show how under increasing noise the system transforms from the noisy quiescence with the small-amplitude stochastic oscilla-

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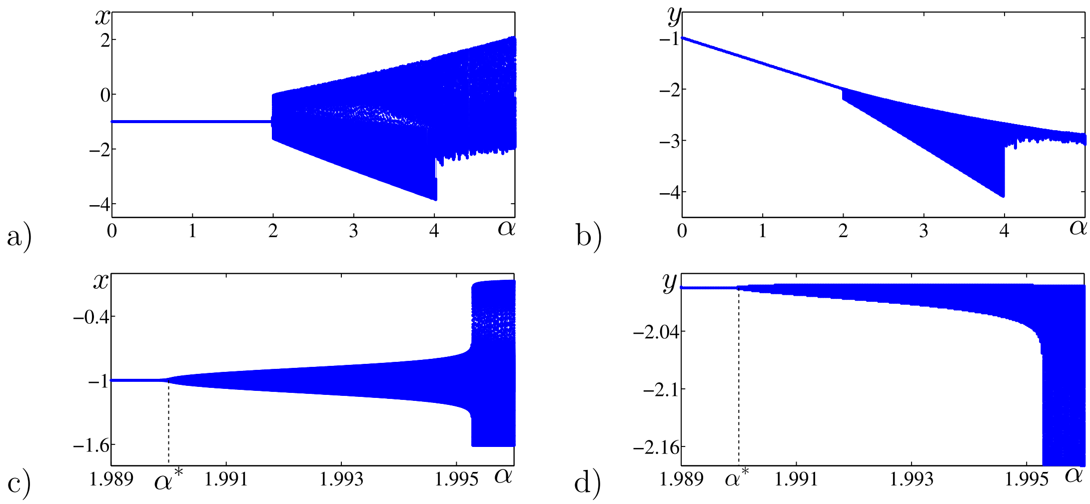

Fig. 1. Bifurcation diagram of the deterministic system (1) with $\sigma = \beta = 0.005$. Enlarged fragments are shown in the bottom panel.

tions to the bursting with large-amplitude stochastic spikes. Here, for the data of the direct numerical simulations, statistics of interspike intervals are analyzed.

In Section 4, for the analysis of the geometric probabilistic mechanisms of the noise-induced generation of the bursting, the stochastic sensitivity function technique and the method of the confidence ellipses is used. We show that the onset of the bursting in the case of stable equilibria can be predicted by the analysis of the mutual arrangement of the confidence ellipses and sub- and superthreshold zones. In Section 5, using largest Lyapunov exponents, we show that the noise-induced transition from the quiescence to stochastic bursting regime is accompanied by the transformation of dynamics from regular to chaotic.

2. Deterministic Rulkov model

Consider the two-dimensional Rulkov model

$$\begin{cases} x_{t+1} = \dfrac{\alpha}{1+{x_t}^2} + y_t \\ y_{t+1} = y_t - \sigma x_t - \beta \end{cases}, \tag{1}$$

where x and y are fast and slow variables, respectively, and the parameters α, σ and β are positive. In what follows, we will fix $\sigma = \beta = 0.005$ and study a behavior of this system in dependence on the parameter α.

The Rulkov model (1) has a unique equilibrium M with coordinates $\bar{x} = -1$, $\bar{y} = -1 - \frac{\alpha}{2}$. For this equilibrium, the Jacobi matrix is

$$J = \begin{pmatrix} \dfrac{\alpha}{2} & 1 \\ -0.005 & 1 \end{pmatrix}.$$

The equilibrium M is stable on the interval $0 < \alpha < 1.99$. The parameter value $\alpha^* = 1.99$ corresponds to the point of the Neimark–Sacker bifurcation with the birth of a closed invariant curve. In Fig. 1, the bifurcation diagram of the deterministic model is presented. Here, x- and y-coordinates of attractors are plotted.

The Fig. 2 shows the phase portraits of the deterministic system (1) for the $\alpha = 1.9$ and $\alpha = 1.98$ in the zone of stable equilibria. As can be seen, if the deviation of the starting point from the equilibrium is small, then the trajectory quickly relaxes to this equilibrium, and a subthreshold response is observed. If the initial deviations is larger than some threshold, the deterministic system exhibits large-amplitude loop before it returns to the small vicinity of the equilibrium and asymptotically tends to M. In this case, the system exhibits a superthreshold response, and a phenomenon of the firing a spike occurs.

Closed invariant curves are shown in Fig. 3 for different values of the parameter $\alpha > \alpha^*$. As can be seen, a size of these closed curves increases when α goes away from the bifurcation point α^*. Note that near $\alpha = 1.995$, a sharp jump of the amplitude values is observed. Such behavior is known as Canard explosion [26]. In the zone of the Canard explosion, both amplitude and the form of these closed invariant curves significantly change. In this regime, system (1) demonstrates a tonic spiking.

In the present paper, we focus on the parameter zone $\alpha < \alpha^*$ and study a response of the equilibria of the Rulkov model to the stochastic forcing.

3. Stochastic excitability of the equilibrium

Consider the stochastically forced two-dimensional Rulkov model

$$\begin{cases} x_{t+1} = \dfrac{\alpha}{1+{x_t}^2} + y_t + \varepsilon_1 \xi_{1,t} \\ y_{t+1} = y_t - \sigma x_t - \beta + \varepsilon_2 \xi_{2,t} \end{cases}, \tag{2}$$

where $\xi_{1,\,t}$, $\xi_{2,\,t}$ are uncorrelated Gaussian random processes with parameters $\mathrm{E}(\xi_{1,t}) = \mathrm{E}(\xi_{2,t}) = 0$, $\mathrm{E}({\xi^2}_{1,t}) = \mathrm{E}({\xi^2}_{2,t}) = 1$, and ε_1, ε_2 are the noise intensities. In what follows, we put $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

Under the influence of random disturbances, solutions of the stochastic system (2) leave the stable equilibrium M and form a regime of stochastic oscillations. For weak noise, random trajectories are localized near M, and the system (2) exhibits small-amplitude stochastic oscillations (see solutions of system (2) with $\alpha = 1.9$, $\varepsilon = 0.0005$ shown by red in Fig. 4(a) and (b).

For larger noise intensities, the solution of system (2) can fall into the superthreshold zone, and as a result the large-amplitude spike is observed. So, the system exhibits the intermittency of the small- and large-amplitude oscillations. This regime can be interpreted as the noise-induced bursting. This type of the behavior is illustrated in Fig. 4(a) and (b) where solutions of system (2) with $\alpha = 1.9$, $\varepsilon = 0.0008$ are shown by blue color. Note that these noise-induced large-amplitude loops are similar in shape to Canard cycles of the deterministic system (compare Figs. 4(a) and 3).

Such changes in dynamics of system (2) lead to the deformation of the probability density function of random states. In Fig. 4(c), for two values of the noise intensity considered above, plots of the probability density function $\rho(x)$ are shown. For weak noise ($\varepsilon =$