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Analysis of 1:4 resonance in a monopoly model with memory

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ABSTRACT

Resonances generate complicated bifurcation sequences. To design a picture of the bifurcation sequence occurring in the presence of a particular strong resonance, the two-parameters bifurcation of the equilibrium of a monopoly model with gradient adjustment mechanism and log-concave demand function is analyzed. Locally, the equilibrium may be destabilized through a period doubling or a supercritical Neimark–Sacker bifurcation. From a global perspective, it is also shown that the model undergoes strong resonances, such as the resonance 1:4, by using a continuation procedure. One of the interesting issue to tackle for this resonance is the investigation of the heteroclinic bifurcations which occur when pairs of saddles form connections near the 1:4 resonance. Therefore, the global analysis, through the description of the phase portraits and the basins of attractions, illustrates the theoretical features associated with the resonance and display interesting and complex dynamical behaviors, like the emergence of *square* and *clover* orbits.

1. Introduction

The rapid development of nonlinear science has greatly promoted the progress in many fields of natural and social disciplines. Nonlinear science has helped to discover and to explain a lot of new phenomena, regulations and inherent complexities in nature and society. During the last decades, the bifurcation problems for dynamical systems have been widely investigated, especially for several models that found applications in various fields like physics, biology, neuroscience, economics, engineering, etc. (see e.g. [1–4] among others for applications to several research fields and [5–11] for a specific focus to economic applications). Generally, these models involve several parameters, but most of the times the bifurcation phenomena are studied as one systemic parameter varies. That is, one-parameter bifurcations are considered. Nonetheless complicated bifurcation structures are likely to take place when more than one parameter of the system is varied at the same time. In particular, if some of the non-degeneracy conditions for the one-parameter bifurcations were violated, two-parameters bifurcations may also occur. Various types of two-parameter bifurcations have been studied in [12–15], just to cite a few. A peculiar aspect in the occurrence of two-parameters bifurcations is related to the existence of strong resonances at which a closed invariant curve might appear in a very peculiar way, or there might be several invariant curves bifurcating from the fixed point. In fact, a Neimark–Sacker bifurcation may take place in resonant cases when the value of a complex eigenvalue $\lambda_1(\varepsilon)$ of a discrete time system at the bifurcation point ε is one of the roots of the equation $\lambda_1^n(\varepsilon) = 1,\ n = 1, 2, 3, 4$.

One of the most intriguing problem in nonlinear dynamical systems (see [27]) is the bifurcation analysis of the orbits near the 1:4 resonance. Several authors analyzed this bifurcation issue, showing rich and various bifurcation situations with complicated and different phase portraits, including precisely homoclinic and heteroclinic connections of different types (see [16–18] among others). It is worth to point out that homoclinic and heteroclinic orbits are of great importance from applied viewpoints (see e.g. [19]). In general, two types of heteroclinic bifurcation of dimension one occur near the 1:4 resonance, the square and the clover connections, the former leaving the attracting cycle outside the connection, while the latter surrounding the stable cycle.

The main purpose of the present paper is to discuss and analyze the bifurcation structures of a two-dimensional discrete time monopoly model. It describes the behavior of a boundedly rational monopolistic firm which faces an inverse demand function and owns constant marginal costs. Since the monopolist does not have a full knowledge of demand, a myopic quantity adjustment mechanism is adopted, in the sense that she/he increases/decreases its own output according to the information given by the marginal profit.

The analysis is carried on by numerical techniques and a continuation procedure, which allow us to detect the presence of a 1:4 resonance within the monopoly model under scrutiny. Due to the difficulties encountered in obtaining rigorous analytical treatment

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of heteroclinic bifurcations near this resonance, by several numerical simulations, we shall show the complete bifurcation structure that takes place moving the parameters around the 1:4 resonance point in order to highlight several dynamic phenomena associated with this codimension-2 bifurcation. We find out that several (global) bifurcations take place around this point and, accordingly, complex dynamics arise associated with fold and Neimark–Sacker bifurcations, homoclinic loop and heteroclinic cycles. The occurrence of 1:4 resonance provides us that, in a certain parameter region, invariant curves may bifurcate from a 4-cycle, meaning that the initially silent model with regular motion would be changed into one that exhibit complex behaviors after a slight change in the relevant parameters. Such complexity may be due to the coexistence of several attractors as well as to the unpredictability of the time path of the system in the long-run due to the presence of chaotic repellors. The latter are associated with homoclinic tangles of the saddle which arise in a certain range of parameters (see e.g. [31]) giving rise to intricate basins structures, while the former relates to the problem of selecting among multiple long-run dynamics, at which the notion of (economic) unpredictability of the final state of the system is connected.

The rest of the paper is organized as follows: Section 2 briefly recalls the main features of the monopoly model and provides the results about the local stability of the fixed point; Section 3 describes the bifurcation sequence associated with the presence of the resonance 1:4 by showing the phase spaces and illustrating the transition between the different dynamic states; Section 4 concludes.

2. A monopoly model with memory and bounded rationality

We briefly recall the monopolistic setting of [21] in which a form of bounded rationality is assumed in the price decision mechanism, and memory is introduced in the formation of price expectations.

The inverse demand function has the general form[1]

$$p = a - b\ln q \tag{1}$$

where p and q denote the price and the quantity of the commodity, a and b are positive parameters.

We consider a linear total cost function which takes the form

$$CT(q) = cq \tag{2}$$

where $c \geq 0$ represents the constant marginal cost.

Due to bounded rationality, the entire demand function is unknown and, accordingly, the monopolist employs a rule of thumb to produce a quantity that guarantees the largest profits. In particular, it is assumed that locally the monopolistic firm, using a gradient mechanism, looks at how the variation of quantity affects the variation of profits. A positive (negative) variation of profits will induce the monopolist to change the quantity in the same (opposite) direction from that of the previous period. No changes will occur if profits are constant. This mechanism can be represented as follows (see e.g. [22,23] among others):

$$q(t+1) = q(t) + \gamma q(t)\frac{\partial \pi(q(t))}{\partial q} \tag{3}$$

where $\gamma > 0$ is the speed of adjustment to misalignments and $\pi(q) = (a - b\ln q)q - cq$ represents the profit function.

As proposed in [21] it may make more sense to assume that the producer considers also the level of the previous production in order to adjust about the future level of realized output. In particular, we will analyze the case in which the monopolist takes into account the most recent production realizations, so that the dynamical system becomes

$$\begin{aligned} q(t+1) &= q(t) + \gamma q(t)\frac{\partial \pi(q^D)}{\partial q} \\ &= q(t) + \gamma q(t)[a - c - b(1 + \ln q^D)] \end{aligned} \tag{4}$$

Assuming

$$q^D = wq(t) + (1-w)q(t-1), \tag{5}$$

i.e. the monopolist employs a weighted average of the two most recent output observations to adjust the future production and $w \in [0, 1]$ represents the weight (memory) given to actual production while $(1-w)$ denotes how much the past output realization is taken into account in forming future output decisions. Substituting (5) into (4) and defining $x(t+1) = q(t)$, the model can be expressed by the following two-dimensional nonlinear map

$$T: \begin{cases} x' = q \\ q' = q + \gamma q[a - c - b(1 + \ln(wq + (1-w)x))] \end{cases} \tag{6}$$

where the superscript $'$ denotes the unit time advancement operator. It is easy to check that the system always admits the unique steady state $Q^* = (q^*, q^*)$, where $q^* = e^{\frac{a-b-c}{b}}$ maximizes the profit function.

We will briefly recall[2] the stability conditions of the steady state: the Jacobian matrix evaluated at Q^* takes the form

$$J_{Q^*} = \begin{pmatrix} 0 & 1 \\ -\gamma b(1-w) & 1 - \gamma bw \end{pmatrix} \tag{7}$$

According to the usual Jury conditions based on the trace and determinant of J_{Q^*} (see e.g. [24]), straightforward computations show that the steady state is locally stable provided that

$$\begin{cases} 2 + \gamma b(1-2w) > 0 \\ 1 - \gamma b(1-w) > 0 \end{cases}$$

In particular, when $\gamma = \gamma_{PD} = \frac{2}{b(2w-1)}$ the steady state undergoes a period-doubling bifurcation while at $\gamma = \gamma_{NS} = \frac{1}{b(1-w)}$ the steady state turns unstable via Neimark–Sacker bifurcation.

The stability region of the steady state is depicted in Fig. 1a by the grey-shaded area in the parameter plane (w, γ), in which the two bifurcation boundaries are highlighted. It is interesting to observe that there exists a double stability threshold for the memory parameter w in which the monopoly equilibrium is locally stable (Fig. 1b provides a bifurcation diagram on varying the memory parameter w and shows the existence of such thresholds); furthermore, as the speed of adjustment γ increases, the stability interval of the steady state Q^* with respect to w is reduced.

Let us now consider the behavior of the system around the curve $\gamma = \gamma_{NS}$. When dealing with a Neimark–Sacker bifurcation curve, we typically have a fixed point with a simple pair of complex-conjugate eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$, which are on the unit circle. In this situation, as it is well explained in [25–27], the center manifold W^c is two-dimensional, and the system on this manifold can be written in complex notations as

$$z \mapsto ze^{\pm i\theta_0}(1 + d_1|z|^2) + O(|z|^4)$$

where $d_1 \in \mathbb{C}$. The non-degeneracy conditions involved are of various types:

1. absence of *strong resonances*:

   $e^{\pm iq\theta_0} \neq 1, \qquad q = 1, 2, 3, 4$

[1] The same specification for the demand function has also been proposed in [20].

[2] We invite the interested reader to [21] for detailed computations.