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# Numerical investigation concerning the dynamics in parameter planes of the Ehrhard–Müller system

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ABSTRACT

In this paper we investigate the nonlinear dynamics of the Ehrhard–Müller system, which is modeled by a set of three-parameter, three autonomous first-order nonlinear ordinary differential equations. More specifically, here we report on numerically computed parameter plane diagrams for this three-parameter system. The dynamical behavior of each point, in each parameter plane, was characterized by using Lyapunov exponents spectra, or independently by counting the number of local maxima of one of the variables, in one complete trajectory in the phase-space. Each of these diagrams indicates parameter values for which chaos or periodicity may be found. In other words, each of these diagrams displays delimited regions of both behaviors, chaos and periodicity. We show that these parameter planes contain self-organized typical periodic structures embedded in a chaotic region. We also show that multistability is present in the Ehrhard–Müller system.



## 1. Introduction

Our motivation in the present work is based on results recently reported by Park and co-workers [1], who have investigated the Ehrhard–Müller system modeled by

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - xz - y + c, \\ \dot{z} &= xy - z, \end{aligned} \tag{1}$$

where $x$, $y$, and $z$ are dynamical variables, and $\sigma$, $r$, and $c$ are parameters that control the dynamics of the system. The nonlinear set of differential equations in (1) was proposed by Ehrhard and Müller [2] to model thermal convection in a single-phase loop with nonsymmetric heating, a more general case than that modeled by the Lorenz system [3]

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - xz - y, \\ \dot{z} &= xy - bz. \end{aligned} \tag{2}$$

By comparing Eqs. (1) and (2), we see that the Ehrhard–Müller system may be thought of as a $b = 1$ Lorenz system, externally excited by a constant forcing $c$.

Some numerical and analytical results were reported in [1]. Equilibrium points were computed, and the related stability analysis was made. Stability regions of equilibrium points were delimited in the $(r, c)$ parameter plane, separated by curves obtained analytically for a fixed parameter $\sigma = 15$. A numerically computed $(r, c)$ isoperiodic diagram was reported, obtained by counting on a grid of equally spaced points, the number of local maxima of the $z$ variable, in one complete trajectory in the phase-space. Additionally, some bifurcation diagrams, attractors in the phase-space, and Lyapunov exponents plots were reported in [1].

In this paper we report on parameter plane plots for the three-parameter, three-dimensional nonlinear dynamical system modeled by Eq. (1), considering all three pairs of parameters, namely $(r,\ c)$, $(r,\ \sigma)$, and $(c,\ \sigma)$. Such diagrams are important because they may be interpreted as cross sections of the $(r, c, \sigma)$ three-dimensional parameter-space of the Ehrhard–Müller system. In addition, parameter plane plots allow us to observe regular (periodic or quasiperiodic) and chaotic orbits on continuous sets of parameters. This knowledge of the organization of chaos and regularity in parameter planes may be useful to choose suitable paths in the parameter-space. For instance, by making appropriate modifications in parameters, we can travel over domains where the system is always chaotic, which is interesting for practical applications involving for example chaos control, chaotic secure communication and chaotic synchronization. More specifically, in this paper we perform a numerical investigation involving the delimitation of stability domains in all three parameter planes of the model (1), to gain information about regions of chaotic and regular behaviors,

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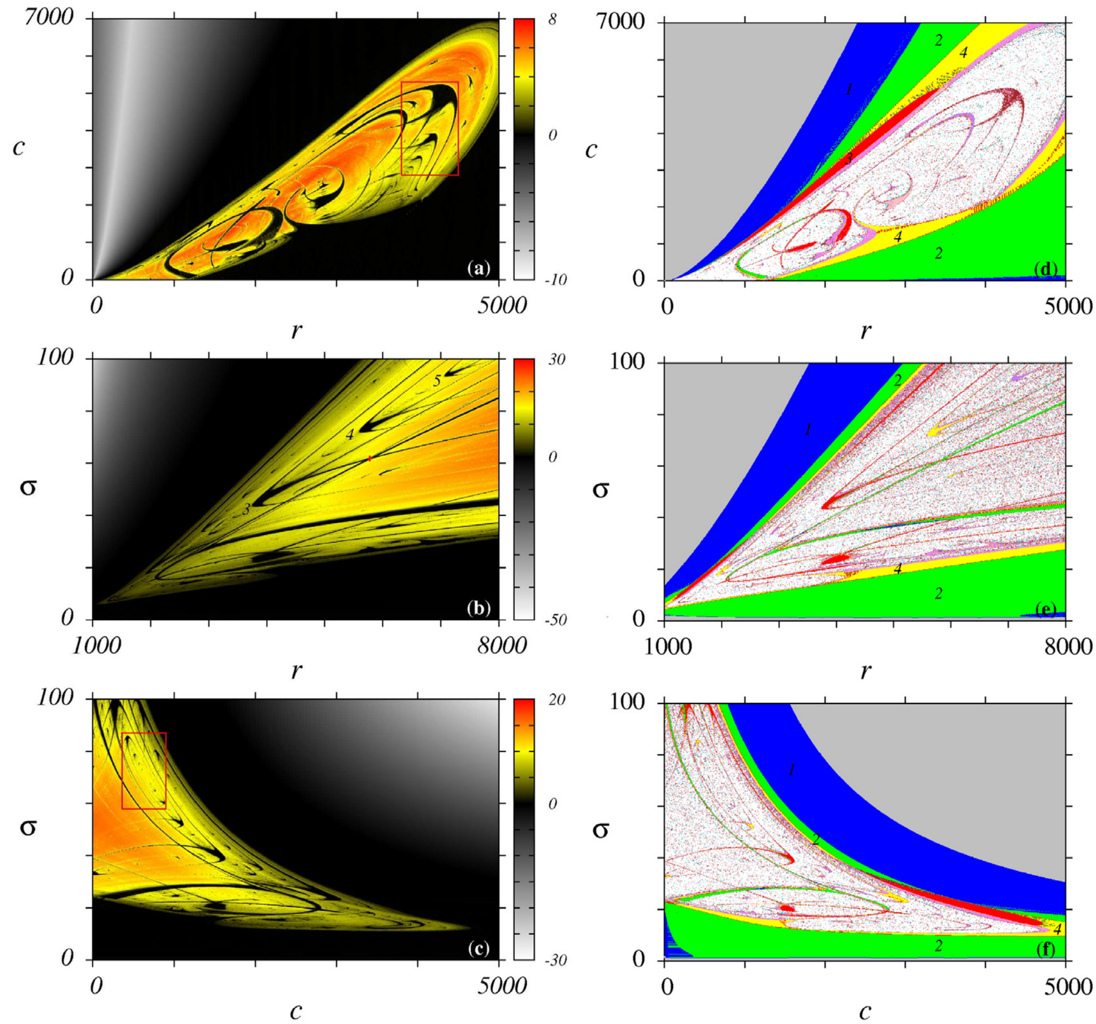


**Fig. 1.** (a) and (d) Global view of the ($r$, $c$) parameter plane for system (1), $0 \le r \le 5000$ and $0 \le c \le 7000$. (b) and (e) Global view of the ($r$, $\sigma$) parameter plane for system (1), $1000 \le r \le 8000$ and $0 \le \sigma \le 100$. (c) and (f) Global view of the ($c$, $\sigma$) parameter plane for system (1), $0 \le c \le 5000$ and $0 \le \sigma \le 100$. In diagrams (a)-(c), color is related to the magnitude of the largest Lyapunov exponent, while in diagrams (d)-(f) number indicates period (see text).

as two of the three parameters in Eq. (1) are simultaneously varied.

The paper is organized as follows. In Section 2 examples of the parameter planes are presented and interpreted, together with other numerical results involving conventional bifurcation diagrams, trajectories in the phase-space (attractors), and basins of attraction. Finally, the paper is summarized in Section 3.

## 2. Some parameter planes of the Ehrhard–Müller system

This section is dedicated to investigate examples of the three different parameter plane diagrams of the model (1), therefore considering all possible combinations of two parameters in a total of three. Each diagram was obtained in a grid of $10^3 \times 10^3$ equally spaced points, by using one of two different methods, to characterize the dynamical behavior of each point in the respective parameter plane. One method considers the largest Lyapunov exponent (LLE), calculated using the Wolf algorithm [4], as the indicator of periodicity or chaos, while the other method considers the number of local maxima of one of the dynamical variables in one complete trajectory in the phase-space, a number that from here on we call period. Except for the two varying parameters in each plot, the other parameter in Eq. (1) was kept fixed as $\sigma = 15$, $r = 3000$, or $c = 2000$, as the case. Regardless of the choice of the two varying parameters, system (1) was integrated by using a fourth order Runge–Kutta algorithm, with a fixed time step size equal to $10^{-4}$, being dropped the first $5 \times 10^5$ integration steps, regarded as a transient. Numerical Integrations were realized starting at the lower value of the two involved parameters, from the initial condition $P_0 = (x_0, y_0, z_0) = (14, 0.5, 1000)$. To begin integrations for each incremented pair of parameters, was used the last value of $P = (x, y, z)$, obtained with the anterior value of the pair of parameters, as the initial condition for the newly incremented pair. In other words, the attractor was followed whenever the parameters were changed, when performing the numerical integration of system (1). With regard to the method that considers the LLE, for the computation of the average involved in the calculation of each one of the $1 \times 10^6$ LLE, we consider the $5 \times 10^5$ integration steps after the transient. Same number of integration steps was considered to determine the $1 \times 10^6$ period values, in the case of the other method. Here we choose the variable $x$ to count the number of local maxima, calling this number (the period) $x_m$.

A global view of the ($r$, $c$), ($r$, $\sigma$), and ($c$, $\sigma$) parameter planes is shown respectively in diagrams (a) and (d), (b) and (e), and (c) and (f), of Fig. 1. Color in diagrams (a)–(c) is related to the magnitude of the respective LLE. A positive LLE is indicated by a continuously changing yellow to red color, a negative LLE is indicated by a continuously changing white to black color, and the black color itself