Chaos, Solitons and Fractals 110 (2018) 169–177

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A numerical approach for solving the fractional Fisher equation using Chebyshev spectral collocation method

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ARTICLE INFO

Article history:
Received 21 October 2017
Revised 24 January 2018
Accepted 13 March 2018

MSC:
41A30
65N12
65N22

Keywords:
Fractional Fisher equation
Chebyshev polynomials
FDM
Convergence analysis

ABSTRACT

In this paper, an efficient numerical method is introduced for solving the fractional (Caputo sense) Fisher equation. This equation presents the problem of biological invasion and occurs, e.g., in ecology, physiology, and in general phase transition problems and others. We use the spectral collocation method which is based upon Chebyshev approximations. The properties of Chebyshev polynomials are utilized to reduce the proposed problem to a system of ODEs, which is solved by using finite difference method (FDM). Some theorems about the convergence analysis are stated. A numerical simulation and a comparison with the previous work are presented. We can apply the proposed method to solve other problems in engineering and physics.

1. Introduction

Fractional derivatives in the sense of the Liouville–Caputo and Riemann–Liouville definitions play an important role in many applications, such as engineering, physical, economic, and biological applications [1–3]. These types with respect to the power law function kernel. Recently, Yang et al. [4] proposed a new definition of the local fractional derivative and they applied it on a various problems in engineering [5]. More recently, Caputo and Fabrizio [6] suggested a new fractional derivative based on the exponential decay law which is a generalized power law function [7–10]. Abdon and Dumitru introduced a fractional derivative with non-local kernel based on the Mittag–Leffler function and permit to describe complex physical problems that follows at the same time the power and exponential decay law [11–13]. So, recently a considerable attention has been given to the solutions of fractional differential equations (FDEs) due to their frequent appearance in various applications in fluid mechanics and viscoelasticity [14,15]. Most FDEs do not have exact solutions, so approximate and numerical techniques must be used [16,17].

Definition 1. The Caputo fractional derivative operator D^{ν} of order ν is defined in the following form:

$$D^{\nu}\varphi(x)=\frac{1}{\Gamma(m-\nu)}\int_0^x\frac{\varphi^{(m)}(\tau)}{(x-\tau)^{\nu-m+1}}d\tau,\quad \nu>0,\quad m-1<\nu\le m,\ m\in\mathbb{N},\ x>0. \tag{1}$$

For more details on fractional derivatives definitions and its properties see [18].

At the last decade, the spectral methods for solving fractional differential equations (FDEs) have attracted the attention of many researchers. Li and Xu [19] developed a hybrid scheme and a time-space spectral method to solve time-fractional diffusion problem; Khader [20] presented a Chebyshev collocation method for solving the space-fractional diffusion equation; while Piret and Hanert developed a radial basis function method for studying fractional diffusion equations [21]. Also, the Legendre spectral method [22] is used to solve the non-linear system of fractional diffusion equations; and an adaptive pseudo spectral method [23]. Also, Khader [24] used the generalized Laguerre spectral algorithms to solve the linear fractional Klein–Gordon equation. In all these methods the

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https://doi.org/10.1016/j.chaos.2018.03.018
0960-0779/

polynomial bases have been used in all spectrum methods. But recently, Zayernouri et al. used the discontinuous spectral element methods to solve the time-and space-fractional advection equations [25] and the fractional delay equations [26]. Spectral collocation method (SCM) is a general approximate analytical method which is used to get the solutions for some of nonlinear differential equations. SCM has some advantages for handling this class of problems in which the Chebyshev coefficients for the solution can be exist very easily after using the numerical programs. For this reason, this method is much faster than the other methods. Chebyshev polynomials are well known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications. They are widely used because of their good properties in the approximation of functions.

The main goal in this article is devoted to study the numerical solution for the following general form of nonlinear fractional differential equation:

$$u_t(x,t) = D^{\alpha}u + N(u), \qquad 0 < x < 1, \quad 0 \le t \le T, \tag{2}$$

where $N(u)$ represents the nonlinear term, and the parameter $0 < \alpha \le 2$ refers to the Caputo fractional order of spatial derivatives. Assume that the following initial and boundary conditions

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{3}$$

$$u(0,t) = g_1(t), \qquad u(1,t) = g_2(t). \tag{4}$$

In our study, we consider the following two cases which depend on the nonlinear term $N(u)$:

Nonlinear fractional Fisher equation (NFFE):

In this case, the nonlinear term $N(u)$ is taken as $N(u) = \mu u(1-u)$, $\mu \in \mathbb{R}$ and the Eq. (2) is called nonlinear fractional Fisher equation. This equation is introduced by Fisher [27] to describe the kinetic advancing rate of an advantageous gene. In a large number of biological and chemical phenomena, the reaction term is represented by $\mu u(1-u)$, where $\mu > 0$ and may be a function in the space variable. Eq. (2) represents the evolution of the population due to the two competing physical processes and changes of interaction of diffusion and nonlinear reaction can be observed. This equation arises in heat and mass transfer, biology, and ecology.

Nonlinear generalized fractional Fisher equation (NGFFE):

In this case, the nonlinear term $N(u)$ is taken as $N(u) = u(1-u)(u-\beta)$, $0 < \beta < 1$ and the Eq. (2) is called nonlinear generalized fractional Fisher equation [28]. This type of equations has travelling wave fronts which appears in the understanding of physical, chemical, and biological phenomena. The wave of advance of advantageous genes in the context of population dynamics is proposed in [28] to describe the spatial spread of an advantageous allele and explored its travelling wave solutions. Also these equations occur, e.g., in ecology, physiology, and in general phase transition problems and others. In case of favourable environmental conditions, the alien population may begin to grow and spread over the area and thus the local initial structural perturbation of the native biological community may lead to large-scale dramatic changes in the community structure [29].

The well-known Fisher's equation combines diffusion with logistic nonlinearity. Fisher proposed Eq. (1) as a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics [30] and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models [31] and branching Brownian motion processes.

The basic aim of this work is to apply the Chebyshev collocation method with the help of finite difference method to discretize the proposed problem (2), greatly simplify this problem to nonlinear systems of algebraic equations which will be solved by using Newton iteration method.

2. Procedure of solution

2.1. An approximate formula for fractional derivatives

Chebyshev polynomials are a family of orthogonal polynomials on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [32]:

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \qquad T_0(z) = 1, \quad T_1(z) = z, \quad n = 1, 2, \dots. \tag{5}$$

In order to use these polynomials on the interval [0,1] we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2t - 1$. The shifted Chebyshev polynomials are defined as $\bar{\mathbb{T}}_n(t) = T_n(2t-1) = T_{2n}(\sqrt{t})$. The analytic form for $\bar{\mathbb{T}}_n(t)$ of degree n is given by

$$\bar{\mathbb{T}}_n(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}\,(n+k-1)!}{(2k)!\,(n-k)!}\,t^k, \qquad n = 2, 3, \dots. \tag{6}$$

The function $\Omega(t) \in L_2[0,\ 1]$ can be expressed as a linear combination of $\bar{\mathbb{T}}_n(t)$ as

$$\Omega(t) = \sum_{i=0}^{\infty} a_i\,\bar{\mathbb{T}}_i(t), \tag{7}$$

where the coefficients a_i are given by

$$a_i = \frac{\rho}{\pi}\int_0^1 \frac{\Omega(t)\,\bar{\mathbb{T}}_i(t)}{\sqrt{1-t^2}}\,dt, \qquad \rho = 1 \ \text{ if } \ i = 0, \quad \rho = 2 \ \text{ if } \ i = 1, 2, \dots. \tag{8}$$

We take the first $(m+1)$-terms of the series (7) to obtain the following approximation form

$$\Omega_m(t) = \sum_{i=0}^{m} a_i\,\bar{\mathbb{T}}_i(t). \tag{9}$$

Theorem 1. *(Chebyshev truncation theorem) [32] If we approximate the function $\Omega(t)$ in the following form*

$$\Omega_m(t) = \sum_{k=0}^{m} a_k\,T_k(t), \tag{10}$$

then the error in this approximation is given by

$$E_T(m) \equiv |\Omega(t) - \Omega_m(t)| \le \sum_{k=m+1}^{\infty} |a_k|, \qquad \forall\,\Omega(t),\ m,\ \ t \in [-1, 1]. \tag{11}$$

The main approximate formula of $D^{\nu}\Omega_m(t)$ is given in the following theorem.

Theorem 2. *[33] Suppose that we approximate the function $\Omega(t)$ in the form (9) then $D^{\nu}(\Omega_m(t))$ can be defined as*

$$D^{\nu}(\Omega_m(t)) = \sum_{i=\lceil\nu\rceil}^{m}\ \sum_{k=0}^{i-\lceil\nu\rceil} a_i\,\Upsilon_{i,k}^{(\nu)}\,t^{i-k-\nu}, \qquad \Upsilon_{i,k}^{(\nu)} = (-1)^{i-k}\frac{2^{2k}\,i\,(i+k-1)!\,\Gamma(k+1)}{(i-k)!\,(2k)!\,\Gamma(k+1-\nu)}. \tag{12}$$

Theorem 3. *[33] The $D^{\nu}(\bar{\mathbb{T}}_i(t))$ can be expressed as a linear combination of the $\bar{\mathbb{T}}_i(t)$ in the following form*

$$D^{\nu}(\bar{\mathbb{T}}_i(t)) = \sum_{k=\lceil\nu\rceil}^{i}\ \sum_{j=0}^{k-\lceil\nu\rceil} \Theta_{i,j,k}\,\bar{\mathbb{T}}_j(t), \tag{13}$$