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Review

Monotone iterative method for the periodic boundary value problems of impulsive evolution equations in Banach spaces☆

Baolin Li*, Haide Gou

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070 PR China

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ABSTRACT

We use a monotone iterative method in the presence of lower and upper solutions to discuss the existence and uniqueness of mild solutions for the boundary value problem of impulsive evolution equation in an ordered Banach space E

$$u'(t)+Au(t)=f(t,u(t),Fu(t),Gu(t)),\quad t\in J,\ t\neq t_k,$$
$$\Delta u|_{t=t_k}=I_k(u(t_k)),\quad k=1,2,\ldots,m,$$
$$u(0)=u(\omega),$$

where A: $D(A)\subset E\to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t\geq 0)$ in E. Under wide monotonicity conditions and the non-compactness measure condition of the nonlinearity f, we obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions requiring only that $-A$ generate a C_0-semigroup.

1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, population dynamics, ecology, chemical, biological systems, and engineering background. Therefore, it has been an object of intensive investigation in recent years; see for example [6]. Consequently, some basic results on impulsive differential equations have been obtained and applications to different areas have been considered by many authors; see [1,4,7,10]. However, the theory still remains to be developed.

In this paper, we use a monotone iterative method in the presence of lower and upper solutions to discuss the existence of mild solutions to the periodic boundary value problem (PBVP) of first order semilinear impulsive integro-differential evolution equations of Volterra type in an ordered Banach space E

$$\begin{aligned} u'(t)+Au(t)&=f(t,u(t),Fu(t),Gu(t)),\quad t\in J,\ t\neq t_k,\\ \Delta u|_{t=t_k}&=I_k(u(t_k)),\quad k=1,2,\ldots,m,\\ u(0)&=u(\omega), \end{aligned}\tag{1.1}$$

where A: $D(A)\subset E\to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t\geq 0)$ in E; $f\in C(J\times E\times E\times E,\ E)$, $J=[0,\omega]$, $\omega>0$ is a constant, $0=t_0<t_1<t_2<\ldots<t_m<t_{m+1}=\omega$; $I_k\in C(E,\ E)$ is an impulsive function, $k=1,2,\ldots,m$; and

$$Fu(t)=\int_0^t K(t,s)u(s)\mathrm{d}s,\ \ K\in C(D,R^+),\tag{1.2}$$

$$Gu(t)=\int_0^\omega H(t,s)u(s)\mathrm{d}s,\ \ H\in C(D_0,R^+),\tag{1.3}$$

where $D=\{(t,s)\in R^2:0\leq s\leq t\leq\omega\}$, $D_0=\{(t,s)\in R^2:0\leq t,s\leq\omega\}$ and $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t=t_k$; i.e., $\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t=t_k$ respectively. Let $PC(J,E)=\{u:J\to E, u(t)$ is continuous at $t\neq t_k$, and left continuous at $t=t_k$, and $u(t_k^+)$ exists, $k=1,2,\ldots,m\}$. Evidently, $PC(J,\ E)$ is a Banach space with the norm $\|u\|_{PC}=\sup_{t\in J}\|u(t)\|$. Denote E_1 by the norm $\|\cdot\|_1=\|\cdot\|+\|A\cdot\|$. Let $J'=J\backslash\{t_1,t_2,\ldots,t_m\}$. An abstract function $u\in PC(J,\ E)\cap C^1(J',\ E)\cap C(J',\ E_1)$ is called a solution of PBVP (1.1) if $u(t)$ satisfies all the equalities of (1.1).

The monotone iterative technique in the presence of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Recently, Du and Laksh-

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* Corresponding author.
E-mail address: ghdzxh@163.com (B. Li).

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mikantham [3], Sun and Zhao [12] investigated the existence of minimal and maximal solutions to initial value problem of ordinary differential equation without impulse by using the method of upper and lower solutions and the monotone iterative technique. Guo and Liu [4] developed the monotone iterative method for impulsive integro-differential equations, and built a monotone iterative method for impulsive ordinary integro-differential equation for the IVP in E

$$\begin{aligned} u'(t) &= f(t,u(t),Fu(t)), \quad t\in J,\ t\neq t_k,\\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k=1,2,\ldots,m,\\ u(0) &= x_0. \end{aligned} \tag{1.4}$$

They proved that if (1.4) has a lower solution v_0 and an upper solution w_0 with $v_0\le w_0$, and the nonlinear term f satisfies the monotonicity condition

$$\begin{aligned} f(t,x_2,y_2)-f(t,x_1,y_1) &\ge -M(x_2-x_1)-M_1(y_2-y_1),\\ v_0(t)\le x_1\le x_2\le w_0(t),\ \ Fv_0(t) &\le y_1\le y_2\le Fw_0(t), \quad \forall t\in J. \end{aligned} \tag{1.5}$$

They also required that the nonlinear term f and the impulsive function I_k satisfy the noncompactness measure condition

$$\alpha(f(t,U,V))\le L_1\alpha(U)+L_2\alpha(V), \tag{1.6}$$

$$\alpha(I_k(D))\le M_k\alpha(D), \quad k=1,2,\ldots,m, \tag{1.7}$$

where U, V, $D\subset E$ are arbitrarily bounded sets, L_1, L_2 and M_k are positive constants and satisfy

$$2a(M+L_1+aK_0L_2)+\sum_{k=1}^{m}M_k<1, \tag{1.8}$$

where $K_0=\max_{(t,s)\in\Delta}K(t,s)$, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E. Then IVP(1.4) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively. Latter, Li and Liu [7] expanded the results in [4], they obtained the existence of the extremal solutions to the initial value problem for impulsive ordinary integro-differential equation (1.4), but did not require the noncompactness measure condition (1.7) for impulsive function I_k and the restriction condition (1.8).

The existence of solutions of impulsive differential equation of the form

$$u'(t)=f(t,u(t),Fu(t),Gu(t)), \quad 0<t<T_0,\ t\neq t_i, \tag{1.9}$$

$$u(0)=u_0, \tag{1.10}$$

$$\Delta u(t_i)=I_i(u(t_i)), \quad i=1,2,3,\ldots,p, \tag{1.11}$$

has been studied by many authors [4,15,17]. In the special case where f is uniformly continuous, Guo and Liu [4] established existence theorems of maximal and minimal solutions for (1.9)–(1.11) with strong conditions. Liu [16] studied the existence of mild solutions of the impulsive evolution equation

$$u'(t)=Au(t)+f(t,u(t)), \quad 0<t<T_0,\ t\neq t_i,$$

where A is the infinitesimal generator of C_0 semigroup with the impulsive condition in (1.10)-(1.4) by using semigroup theory.

On the other hand, some authors consider the initial value problem of evolution equations, see [1,8–10,13,14] and the reference therein. But they all require the semigroup $T(t)(t\ge 0)$ generated by $-A$ be equicontinuous semigroup, this is a very strong assumption. In this paper, we will study the boundary value problem of impulsive integro-differential evolution equation (1.1) not requiring the equicontinuity of the semigroup $T(t)(t\ge 0)$ generated by $-A$. We obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions only requiring the semigroup $T(t)(t\ge 0)$ generated by $-A$ is a C_0-semigroup in E.

The paper is organized as follows: In Section 2 we recall some basic known results and introduce some notations. In Section 3 we discuss the existence theorem for boundary value problem (1.1). An example will be presented in Section 4 illustrating our results.

2. Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel.

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\le$, whose positive cone $P=\{x\in E: x\ge 0\}$ is normal with normal constant N. Let $C(J, E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C=\max_{t\in J}\|u(t)\|$. Evidently, $C(J, E)$ is also an ordered Banach space reduced by the convex cone $P'=\{u\in E|u(t)\ge 0, t\in J\}$, and P' is also a normal cone. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [2]. For any $B\subset C(J, E)$ and $t\in J$, set $B(t)=\{u(t): u\in B\}\subset E$. If B is bounded in $C(J, E)$, then $B(t)$ is bounded in E, and $\alpha(B(t))\le\alpha(B)$.

We first give the following lemmas to be used in proving our main results.

Lemma 2.1 ([5]). *Let $B=\{u_n\}\subset PC(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt : n\in\mathbb{N}\Big\}\Big)\le 2\int_J\alpha(B(t))dt.$$

Let A: $D(A)\subset E\to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t\ge 0)$ in E. Then there exist constants $C>0$ and $\delta\in\mathbb{R}$ such that

$$\|T(t)\|\le Ce^{\delta t}, \quad t\ge 0.$$

Definition 2.1. A C_0-semigroup $T(t)(t\ge 0)$ is said to be exponentially stable in E if there exist constants $C\ge 1$ and $\delta>0$ such that

$$\|T(t)\|\le Ce^{-\delta t}, \quad t\ge 0.$$

It is well-known [11], Chapter 4, Theorem 2.9 that for any $x_0\in D(A)\subset E$ and $h\in C(J, E)$, the initial value problem of the linear evolution equation

$$\begin{aligned} u'(t)+Au(t) &= h(t), \quad t\in J,\\ u(t_0) &= x_0, \end{aligned} \tag{2.1}$$

has a unique mild solution $u\in C(J, E)$ given by

$$u(t)=T(t-t_0)x_0+\int_{t_0}^{t}T(t-s)h(s)ds, \quad t\in J. \tag{2.2}$$

To prove our main results, for any $h\in PC(J, E)$, we consider the periodic boundary value problem (PBVP) of linear impulsive evolution equation in E

$$\begin{aligned} u'(t)+Au(t) &= h(t), \quad t\in J',\\ \Delta u|_{t=t_k} &= y_k, \quad k=1,2,\ldots,m,\\ u(0) &= u(\omega), \end{aligned} \tag{2.3}$$

where $y_k\in E$, $k=1,2,\ldots,m$, $x_0\in E$.

Lemma 2.2. *Let $T(t)(t\ge 0)$ be an exponentially stable C_0-semigroup in E generated by $-A$, for any $h\in PC(J, E)$, $x_0\in E$ and $y_k\in E$, $k=1,2,\ldots,m$, then the linear PBVP (2.3) has a unique mild solution*