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On the laminar boundary-layer flow over rotating spheroids

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ABSTRACT

We study the laminar boundary-layer flow over a general spheroid rotating in otherwise still fluid. In particular, we distinguish between prolate and oblate spheroids and use an appropriate spheroidal coordinate system in each case. An eccentricity parameter e is used to distinguish particular bodies within the oblate or prolate families and the laminar-flow equations are established for each family with e as a parameter. In each case, setting $e = 0$ reduces the equations to those already established in the literature for the rotating sphere. We begin by solving the laminar-flow equations at each latitude using a series-solution approximation. A comparison is then made to solutions obtained from an accurate numerical method. The two solutions are found to agree well for a large range of latitudes and eccentricities, and at these locations the series solution is to be preferred due to its simplicity and ease of computation. A discussion of the resulting flows is given with particular emphasis on the implications for their hydrodynamic stability. Their stability characteristics are expected to be very similar to those over the rotating sphere as already studied in the literature.

1. Introduction

The continuous development of spinning projectiles and other industrial applications has led to the need to understand the laminar boundary-layer flow and subsequent onset of transition over the general family of rotating spheroids. Clearly a method of accurate computation of the laminar flow at various locations over the rotating spheroid is the first stage in any such investigation, and this is presented here. An investigation of the convective and absolute instability properties of the laminar flows obtained will be presented in later publications.

Garrett and Peake's [1–3] related stability analyses of the rotating-sphere boundary layer began by first computing the laminar flow profiles using the series-solution method due to Howarth [4], Nigham [5] and Banks [6] and then proceeded by using a more accurate numerical solution. It is therefore natural that we proceed in a similar way.

To our knowledge the only published work on the laminar boundary layer over a rotating spheroid is due to Fadnis [7] who extended the Nigham series solution for the rotating sphere. However, Banks has since showed a flaw in Nigham's solution and this follows through into Fadnis's work. Indeed, the formulation used is such that the results cannot be verified against the laminar-flow profiles already established for a rotating sphere, which is a particular case of spheroid.

In Section 2 we formulate the governing partial differential equations (PDEs) for the laminar boundary-layer flow over rotating spheroids. Distinct coordinate systems are used for each spheroidal family (prolate and oblate) and we define an eccentricity parameter to distinguish particular bodies within each family. In Section 3.1 the governing PDEs are solved using an extension of the method originally developed by Banks for the rotating sphere. The resulting flow profiles are compared

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with direct numerical solutions of the PDEs obtained using a commercially available routine in Section 3.2. A discussion of the resulting flows is given in Section 5 with particular emphasis on the implications for their hydrodynamic stability.

Unfortunately there are no other experimental or theoretical studies of rotating-spheroid boundary layers in the literature and we are therefore unable to verify our results directly. However, our formulation is consistent with existing investigations into the rotating sphere and disk boundary layers and we find favorable comparisons between those and our results when appropriate parameter values are used.

2. Formulation

We formulate the steady laminar-flow equations for each spheroidal family using a distinct coordinate system. In each case a Cartesian frame of reference is used that is fixed in space and has origin located at the center of the body. The spheroid rotates with constant angular velocity $\Omega^\star$ about the z-axis. The quantity $\eta^\star$ is then the distance from the origin and normal to the body surface at a particular latitude θ and azimuth ϕ. Furthermore, $d^\star$ is the distance of each focus from the origin. Note that an asterisk denotes a dimensional quantity.

For the prolate spheroid we use a prolate spheroidal coordinate system defined relative to the Cartesian coordinates as

$$x^\star = \sqrt{{\eta^\star}^2 - {d^\star}^2}\sin\theta\cos\phi, \quad y^\star = \sqrt{{\eta^\star}^2 - {d^\star}^2}\sin\theta\sin\phi, \quad z^\star = \eta^\star\cos\theta.$$

Similarly, for the oblate spheroid we use an oblate spheroidal coordinate system defined as

$$x^\star = \eta^\star\sin\theta\cos\phi, \quad y^\star = \eta^\star\sin\theta\sin\phi, \quad z^\star = \sqrt{{\eta^\star}^2 - {d^\star}^2}\cos\theta.$$

In both cases we consider the hemispheroid defined by rotation of the region $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \phi \leqslant 2\pi$ about the z-axis, with surface given by $\eta_0^\star(\theta,\phi)$ such that the spheroid nose is at $\eta_0^\star(0,\phi)$. Both coordinate systems are consistent with those discussed by Morse [8] and we have confirmed that each system $(\eta^\star,\theta,\phi)$ is orthogonal and reduces to the spherical coordinate system as $d^\star \to 0$.

We introduce $e = d^\star/\eta_{\mathrm{m}}^\star \in [0,1]$, which defines the constant eccentricity of the cross-sectional ellipse. The quantity $\eta_{\mathrm{m}}^\star$ is the length of the semi-major axis of the spheroid which is along the axis of rotation for the prolate spheroid and perpendicular to it in the oblate case. The Navier–Stokes equations are transformed to either coordinate system with some manipulation using the transformations defined above and both sets of equations reduce to those in spherical coordinates for $e = 0$. We apply Prandtl's boundary-layer assumptions to obtain the dimensional boundary-layer equations that govern the laminar flow.

In order to obtain the non-dimensional boundary-layer equations we scale the velocities on the equatorial surface speed of the spheroid, as in Eq. (1). This is consistent with Garrett and Peake's formulation of the rotating sphere

$$U = \frac{U^\star}{\Omega^\star a^\star}, \quad V = \frac{V^\star}{\Omega^\star a^\star}, \quad W = \frac{W^\star}{(\nu^\star\Omega^\star)^{1/2}}. \tag{1}$$

Here $U(\eta,\theta;e)$, $V(\eta,\theta;e)$ and $W(\eta,\theta;e)$ are the scaled velocities in the θ-, ϕ- and η-directions, respectively. Note that $a^\star$ is the equatorial radius of the body defined separately for each spheroid: for the prolate case $a^\star = \eta_{\mathrm{m}}^\star\sqrt{1-e^2}$ and for the oblate case $a^\star = \eta_{\mathrm{m}}^\star$. Further, η is the distance in the normal direction from the surface of the spheroid, scaled on the boundary-layer thickness $\delta^\star = (\nu^\star/\Omega^\star)^{1/2}$, such that $\eta = (\Omega^\star/\nu^\star)^{1/2}(\eta^\star - \eta_0^\star)$, where $\nu^\star$ is the coefficient of kinematic viscosity. Since we are considering spheroids that rotate within otherwise still fluids, the mean pressure $P^\star$ is constant and can be neglected in this analysis.

For the prolate family, the resulting laminar-flow equations are

$$W\frac{\partial U}{\partial\eta} + U\frac{\partial U}{\partial\theta} - V^2\cot\theta = \sqrt{\frac{1-e^2}{1-e^2\cos^2\theta}}\frac{\partial^2 U}{\partial\eta^2}, \tag{2}$$

$$W\frac{\partial V}{\partial\eta} + U\frac{\partial V}{\partial\theta} + UV\cot\theta = \sqrt{\frac{1-e^2}{1-e^2\cos^2\theta}}\frac{\partial^2 V}{\partial\eta^2}, \tag{3}$$

$$\frac{\partial W}{\partial\eta} + \frac{\partial U}{\partial\theta} + \left(\frac{e^2\cos\theta\sin\theta}{1-e^2\cos^2\theta} + \cot\theta\right)U = 0, \tag{4}$$

and for the oblate family

$$W\frac{\partial U}{\partial\eta} + \frac{1}{\sqrt{1-e^2}}\left(U\frac{\partial U}{\partial\theta} - V^2\cot\theta\right) = \sqrt{\frac{1-e^2}{1-e^2\sin^2\theta}}\frac{\partial^2 U}{\partial\eta^2}, \tag{5}$$

$$W\frac{\partial V}{\partial\eta} + \frac{1}{\sqrt{1-e^2}}\left(U\frac{\partial V}{\partial\theta} + UV\cot\theta\right) = \sqrt{\frac{1-e^2}{1-e^2\sin^2\theta}}\frac{\partial^2 V}{\partial\eta^2}, \tag{6}$$

$$\sqrt{1-e^2}\frac{\partial W}{\partial\eta} + \frac{\partial U}{\partial\theta} + \left(\cot\theta - \frac{e^2\cos\theta\sin\theta}{1-e^2\sin^2\theta}\right)U = 0. \tag{7}$$