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# An improved meshless algorithm for a kind of fractional cable problem with error estimate

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ABSTRACT

The present paper is devoted to the development of a kind of spectral meshless radial point interpolation (SMRPI) technique for solving fractional cable equation in one and two dimensional cases. The time fractional derivative is described in the Riemann–Liouville sense. The applied approach is based on a combination of meshless methods and spectral collocation techniques. The point interpolation method with the help of radial basis functions is used to construct shape functions which act as basis functions in the frame of SMRPI. It is proved the scheme is unconditional stable with respect to the time variable in $H^1$ and convergent by the order of convergence $\mathcal{O}(\delta t^{\gamma})$, $0<\gamma<1$. In the current work, the thin plate splines (TPS) are used as the basis functions. The results of numerical experiments are compared with analytical solution to confirm the accuracy and efficiency of the presented scheme.



## 1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [1–3]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [4]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [1], the book of Podlubny [2] and the papers of Metzler and Klafter [5], Bagley and Trovik [6]. Also a comprehensive overview of the development history of fractional calculus has been given in [7].

As is said in [3] the cable equation is one of the most fundamental equations for modeling neuronal dynamics. The Nernst–Planck equation of electrodiffusion for the movement of ions in neurons has also been shown to be equivalent to the cable equation under simplifying assumptions [8]. Some authors have elucidated that if the ions are undergoing anomalous subdiffusion then the comparison with models that assume standard or normal diffusion will likely lead to incorrect or misleading diffusion coefficient values [9], and models that incorporate anomalous diffusion should be used. Langlands et al. [10,11] derived a fractional variant of the Nernst–Planck equation to model the anomalous subdiffusion of the ions. The present paper considers one and two-dimensional fractional cable equation as follows

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} = \rho_1\left[{}_0D_t^{1-\alpha}\Delta u(\mathbf{x},t)\right] - \rho_2\left[{}_0D_t^{1-\beta}u(\mathbf{x},t)\right] + f(\mathbf{x},t), \\ \mathbf{x}\in\Omega\subset\mathbb{R}^d, d=1,2, \qquad t\in(0,T], \end{cases} \tag{1}$$

with initial condition

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x}\in\overline{\Omega} = \Omega\cup\partial\Omega, \tag{2}$$

and boundary condition

$$u(\mathbf{x},t) = h(\mathbf{x},t), \qquad \mathbf{x}\in\partial\Omega, t>0, \tag{3}$$

where $\Delta$ is Laplacian operator, $\partial\Omega$ is the closed curve bounding the region and $\overline{\Omega}$ denotes the spatial domain, $0<\alpha,\ \beta<1$, $\rho_1$ and $\rho_2$ are two positive constants, and $u_0(\mathbf{x},t)$, $h(\mathbf{x},t)$ are known sufficiently smooth functions in their respective domains. The notation ${}_0D_t^{1-\gamma}$ $(\gamma=\alpha,\beta)$ denotes the Riemann–Liouville fractional derivative operator which is defined as

$${}_0D_t^{1-\gamma}u(\mathbf{x},t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{u(\mathbf{x},s)}{(t-s)^{1-\gamma}}ds, \qquad \gamma=\alpha,\beta, \tag{4}$$

where $\Gamma(.)$ is the Gamma function. Some different works have been done on developing numerical methods for solving the fractional cable equation [12,13]. As a non-complete list, one can be referred to the resources that will be coming. Henry et al. [14] studied the fractional Nernst–Planck equation for modeling anomalous electrodiffusion in spiny dendrites. They subsequently found

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a fractional cable equation by treating the neuron and its membrane as two separate materials governed by separate fractional Nernst–Planck equations and employed a small ionic concentration gradient assumption [10,11]. The resulting equation involves two fractional temporal Riemann–Liouville derivatives. Langlands et al. [15] modeled the anomalous subdiffusion by replacing diffusion constants with time dependent operators parameterized by fractional order exponents. They obtained fundamental solutions of the fractional cable equations as functions of the scaling parameters for infinite cables and semi-infinite cables with instantaneous current injections. Also, they derived action potential firing rates based on simple integrate and fire versions of the models. The same authors in [16] presented solutions on finite domains for mixed Robin boundary conditions. Liu et al. [3] proposed two new implicit numerical methods with convergence order $\mathcal{O}(\delta t+h^2)$ and $\mathcal{O}(\delta t^2+h^2)$ for the fractional cable equation, where $\delta t$ and $h$ are the time and space step sizes. They investigated the stability and convergence of these methods using the energy method. Lin et al. [17] proposed a schema combining a finite difference approach in the time direction and a spectral method in the space direction and analyzed for the fractional cable equation. Also they proved unconditional stability and convergence of the method. Hu and Zhang [18] proposed two implicit compact difference schemes for the fractional cable equation and in addition, they proved the stability and convergence in $L_\infty$-norm of these methods by the energy method. Zhang et al. [19] proposed discrete-time orthogonal spline collocation (OSC) methods for the two-dimensional fractional cable equation. They proved unconditional stability and convergence with the order $\mathcal{O}(\delta t^{\min(2-\gamma_1,2-\gamma_2)}+h^{r+1})$ in $L^2$-norm where $\delta t$, $h$ and $r$ are the time step size, space step size and polynomial degree, respectively, and $\gamma_1$ and $\gamma_2$ are two different exponents of fractional derivatives with $0<\gamma_1,\ \gamma_2<1$. Dehghan and Abbaszadeh [20] proposed the element free Galerkin technique for the fractional cable equation. Moreover, stability and convergence of this method with convergence orders of the time discrete scheme and the full discrete scheme were discussed. They changed the main problem with Dirichlet boundary condition to a new problem with Robin boundary condition. Then they show convergence orders of the time discrete scheme and the full discrete scheme are $\mathcal{O}(\delta t^{1+min\{\alpha,\beta\}})$ and $\mathcal{O}(r^{p+1}+\delta t^{1+min\{\alpha,\beta\}})$, respectively. To see more references in the field of fractional cable equation refer to Section 1 of Ref. [20].

The main shortcoming of mesh-based methods such as the finite element method (FEM) [21], the finite volume method (FVM) [22] and the boundary element method (BEM) [23] is that these numerical methods rely on meshes or elements. In the two last decades, in order to overcome the mentioned difficulties some techniques so-called meshless methods have been proposed. A brief review of the meshless method has been studied in [24].

In spite of great benefits in using the meshless weak form methods, there are some limitations. For example, the complicated nature of the non-polynomial shape functions may be computationally expensive to implement in a numerical integration scheme. On the other hand, some methods such as those that are based on moving least squares (MLS) and RBFs, need to determine a shape parameter which plays the important role in the accuracy of the methods. Furthermore, the resultant linear systems might be ill-conditioned and to overcome this defect, some regularization methods are needed. In the meshless method based on strong form, such as Kansa's method, this RBF collocation approach is inherently meshless, easy-to-program, and mathematically very simple to learn, but its fundamental flaw is un-stability because of the use of the global strong form. To overcome these shortages, we propose a new spectral meshless radial point interpolation (SMRPI) method which is based on meshless radial point interpolation and spectral collocation techniques [25–27]. In the SMRPI method, the point interpolation method by the help of radial basis functions is proposed to construct shape functions which have Kronecker delta function property and are used as basis functions in the frame of the SMRPI. Based on the spectral methods, evaluation of high-order derivatives of given differential equation is easy by constructing and using operational matrices. The SMRPI method does not require any kind of integration locally over small quadrature domains nor regularization techniques. Therefore, the computational cost of the SMRPI method is less expensive.

Our aim in this work is the development of SMRPI method to obtain the solution of fractional cable equation with the details as follows. In Section 2, we obtain a time discrete scheme to handle Eq. (1). In this section, we also prove the unconditional stability and convergence of the time discrete scheme and prove that computational order of time discrete scheme is $\mathcal{O}(\delta t^\gamma)$, $0<\gamma<1$. In Section 3, we introduce the SMRPI scheme and obtain the shape functions in SMRPI. Time discretization approximation for implementation of the SMRPI is given in Section 4. In Section 5, we report the numerical experiments of solving Eq. (1) for three test problems. Finally a conclusion is given in Section 6.

## 2. The time discretization approximation

To introduce a finite difference approximation in order to discretize the time-fractional derivative, we need some preliminaries. Let us define

$$t_n=n\delta t \qquad n=0,1,2,\dots,K,$$

where $\delta t=T/K$ is the step size of time variable. The notation $I^\gamma_{a+}y(t)$ for $y(t)\in L^1(a,\ b)$ denotes the Riemann–Liouville fractional integral operator which is defined as

$$I^\gamma_{a+}y(t)=\frac{1}{\Gamma(\gamma)}\int_a^t\frac{y(s)}{(t-s)^{1-\gamma}}ds, \qquad t>a,\gamma>0, \tag{5}$$

**Lemma 1.** *(Zhuang and Liu [28]) If $y(t)\in C^2[0,\ T]$, then*

$$I^\gamma_{0+}y(t_{n+1})-I^\gamma_{0+}y(t_n)=\frac{\delta t^\gamma}{\Gamma(\gamma+1)}\Big(y(t_{n+1})+\sum_{j=0}^{n-1}(w_{j+1}-w_j)y(t_{n-j})\Big)+R_{n,\gamma}, \tag{6}$$

*in which*

$$|R_{n,\gamma}|\le Cw_n\delta t^{1+\gamma},\quad 1=w_0>w_1>\ldots>w_n>0 \ \text{ and } \ w_j=(j+1)^\gamma-j^\gamma.$$

Using the above notations, we have

$${}_0D_t^{1-\gamma}u(\mathbf{x},t)=\frac{\partial}{\partial t}I^\gamma_{0+}u(\mathbf{x},t). \tag{7}$$

We discretize the time variable using forward finite difference relation for approximating the first-order derivative on time variable. So, Eq. (1) is written as

$$\frac{u(\mathbf{x},t_{n+1})-u(\mathbf{x},t_n)}{\delta t}=\frac{\rho_1}{\delta t}\Big(I^\alpha_{0+}\Delta u(\mathbf{x},t_{n+1})-I^\alpha_{0+}\Delta u(\mathbf{x},t_n)\Big)-\frac{\rho_2}{\delta t}\Big(I^\beta_{0+}u(\mathbf{x},t_{n+1})-I^\beta_{0+}u(\mathbf{x},t_n)\Big)+f(\mathbf{x},t_{n+1}). \tag{8}$$

Now from Lemma 1, we can conclude

$$\big(1+\mu_2\big)u^{n+1}(\mathbf{x})-\mu_1\quad\Delta u^{n+1}(\mathbf{x})=u^n(\mathbf{x})+\mu_1\sum_{j=0}^{n-1}\big(\lambda_{j+1}-\lambda_j\big)\Delta u^{n-j}(\mathbf{x})$$