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Dynamical analysis of a fractional-order Rosenzweig–MacArthur model incorporating a prey refuge

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ABSTRACT

This paper considers a fractional order Rosenzweig-MacArthur (R-M) model incorporating a prey refuge. The model is constructed and analyzed in detail. The existence, uniqueness, non-negativity and boundedness of the solutions as well as the local and global asymptotic stability of the equilibrium points are studied. Sufficient conditions for the stability and the occurrence of Hopf bifurcation for the fractional order R-M model are demonstrated. The resolution of the paradox of enrichment is investigated. The impact of fractional order and the prey refuge effects on the stability of the system are also studied both theoretically and by using numerical simulations.

1. Introduction

The dynamics of prey-predator systems are active research topics in ecology and mathematical biology. One focus area is the study the persistence and stability of these systems [1]. Prey can move to areas called refuges where they are safe from their predators and this behaviour may reduce the possibility of prey mortality [2]. Incorporating a refuge is believed to provide a more realistic prey-predator model i.e. for a number of prey populations some form of refuge in the ecosystem is available. Some studies of the dynamical behaviour of prey-predator models incorporating refuge include [1,3–10].

After killing a prey, a predator typically eats and digests its captured food. Some models assume that this occurs at a constant rate [11,12]. The Rosenzweig–MacArthur (R-M) model [11,12] is based on the assumption that the eating and digesting process occurs at a non-constant rate. Studies on the R-M model include [9,13–15]. A R-M model normally incorporates the Holling type-II functional response. The Holling type-II functional response is a type of function in which the attack rate of predator increases at a decreasing rate with prey density until it becomes constant due to satiation [16]. Rosenzweig [17,18] highlighted that increasing the carrying capacity of the prey may lead to an extinction of species in the ecosystem. This is known as the paradox of enrichment.

Kar [15], considered a prey-predator model incorporating a prey refuge, which employs a R-M model with Holling type-II functional response, as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)xy}{1+a(1-m)x}, \\ \frac{dy}{dt} &= \frac{c\beta(1-m)xy}{1+a(1-m)x}-\gamma y. \end{aligned} \tag{1}$$

All the parameters are non-negative for all time $t \geq 0$. The parameters are described in Table 1.

In recent years, fractional-order differential equations have attracted the attention of researchers due to their ability to provide a good description of certain non-linear phenomena [19]. The fractional order differential equations are generalizations of ordinary differential equations to arbitrary (non-integer) orders. Some researchers studied the fractional order differential equations to describe complex systems in different branches of physics, chemistry and engineering [20]. In the last few years, many researchers have also employed fractional-order biological models [5,14,21–27]. This is because fractional-order differential equations are naturally related to systems with memory [5]. Many biological systems possesses memory and the conception of fractional-order system may be closer to real life situations than integer-order systems. The advantages of fractional-order systems are that they describe the whole time domain for physical processes, while the integer-order model is related to the local properties of a certain position, and

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Table 1
Parameters table for the R-M model.

Parameter	Description
x	Prey population.
y	Predator population.
r	Natural growth rate of the prey.
k	Carrying capacity of the prey.
γ	Death rate of the predator.
c	Maximum value of per capita reduction rate of x due to y.
β	Attack rate.
a	Half saturation constant.
mx	Refuge protecting of the prey.
$(1-m)x$	Prey available to the predator.
$\frac{\beta x}{1+ax}$	Holling type-II functional response.

they allow greater degrees of freedom in the model [28]. In [5] a fractional order predator-prey model with refuge was proposed and issues related to existence, uniqueness, non-negativity, equilibrium points and global stability was studied. However, the paper did not deal with R-M model as such. In this paper, we study a fractional order R-M model by extending the integer order model (1) as follows:

$$\begin{aligned} {}^cD^\alpha x(t) &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)xy}{1+a(1-m)x},\\ {}^cD^\alpha y(t) &= \frac{c\beta(1-m)xy}{1+a(1-m)x}-\gamma y, \end{aligned} \tag{2}$$

with the initial conditions

$$x(0)=x_0>0 \text{ and } y(0)=y_0>0,$$

where $\alpha\in(0,1)$, $m\in[0,1)$ and ${}^cD^\alpha$ is the standard Caputo differentiation. All the parameters of fractional order system (2) are non-negative for all time $t\geq 0$. The Caputo fractional derivative of order α is defined as [19,29]:

$${}^cD^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds, \quad n-1<\alpha<n, \ n\in\mathbb{N}.$$

To the best of the authors' knowledge, the dynamical analysis of a fractional-order Rosenzweig–MacArthur model incorporating a prey refuge has not been performed before. Motivated by these observations, a fractional-order Rosenzweig–MacArthur model incorporating a prey refuge is proposed, and then the qualitative behavior of the model (2) is analysed. The local and global stability of the equilibrium points of the fractional order system (2) are investigated as well as the emergence of Hopf bifurcation in the fractional order system (2) is illustrated. We show that introducing the fractional order to Rosenzweig–MacArthur model resolves the paradox of enrichment. Moreover, a suitable method introduced by Adams-Bashforth-Moulton is applied for the numerical simulation of the fractional-order system (2) to confirm the theoretical results. The numerical simulations focus on the influences of fractional order α and prey refuge on the population densities of both prey and predator. It has been shown that the dynamics of fractional order model (2) is more stable than its integer counterpart (1) because of the domain of stability in the fractional order model (2) is larger than the corresponding domain for integer order model (1).

The organization of this paper is as follows. In Section 2, the existence, uniqueness, non-negativity, boundedness, stability analysis and Hopf bifurcations of fractional order system (2) are presented. In Section 3, the numerical simulations are provided to verify the theoretical results of fractional order system (2). Finally, a brief conclusion of our study is given in Section 4.

2. Analysis

This section studies the existence, uniqueness, non-negativity and boundedness of the solutions of a fractional order system (2). In addition, the stability analysis and Hopf bifurcations of fractional order system (2) are also performed.

2.1. Existence and uniqueness

The sufficient condition for existence and uniqueness of the solution of a fractional order system (2) are as follows:

Theorem 1. *For each non-negative initial conditions, there exists a unique solution of fractional order system (2).*

Proof. We seek a sufficient condition for existence and uniqueness of the solutions of fractional order system (2) in the region $\Psi\times(0,T]$ where

$$\Psi=\{(x,y)\in\mathbb{R}^2 : \max(|x|,|y|)\leq\eta\}.$$

The approach used in [5] is adopted. Consider a mapping $G(X)=(G_1(X),G_2(X))$ and

$$\begin{aligned} G_1(X) &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)xy}{1+a(1-m)x},\\ G_2(X) &= \frac{c\beta(1-m)xy}{1+a(1-m)x}-\gamma y. \end{aligned} \tag{3}$$

For any $X,\bar{X}\in\Psi$, it follows from (3) that

$$\begin{aligned} &\|G(X)-G(\bar{X})\|\\ &=\left|G_1(X)-G_1(\bar{X})\right|+\left|G_2(X)-G_2(\bar{X})\right|\\ &=\left|rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)xy}{1+a(1-m)x}-r\bar{x}\left(1-\frac{\bar{x}}{k}\right)+\frac{\beta(1-m)\bar{x}\bar{y}}{1+a(1-m)\bar{x}}\right|\\ &\quad+\left|\frac{c\beta(1-m)xy}{1+a(1-m)x}-\gamma y-\frac{c\beta(1-m)\bar{x}\bar{y}}{1+a(1-m)\bar{x}}+\gamma\bar{y}\right|\\ &=\left|r(x-\bar{x})-\frac{r}{k}(x^2-\bar{x}^2)\right.\\ &\quad\left.-\frac{\beta(1-m)(xy+a(1-m)x\bar{x}y-\bar{x}\bar{y}-a(1-m)x\bar{x}\bar{y})}{(1+a(1-m)x)(1+a(1-m)\bar{x})}\right|\\ &\quad+\left|\frac{c\beta(1-m)(xy+a(1-m)x\bar{x}y-\bar{x}\bar{y}-a(1-m)x\bar{x}\bar{y})}{(1+a(1-m)x)(1+a(1-m)\bar{x})}\right.\\ &\quad\left.-\gamma(y-\bar{y})\right|\\ &=r|x-\bar{x}|+\frac{r}{k}|(x-\bar{x})(x+\bar{x})|+\gamma|y-\bar{y}|\\ &\quad+\frac{(1+c)a\beta(1-m)^2x\bar{x}|y-\bar{y}|}{(1+a(1-m)x)(1+a(1-m)\bar{x})}\\ &\quad+\frac{(1+c)\beta(1-m)|xy-\bar{x}y+\bar{x}y-\bar{x}\bar{y}|}{(1+a(1-m)x)(1+a(1-m)\bar{x})}\\ &\leq r|x-\bar{x}|+\frac{2\eta r}{k}|x-\bar{x}|+\gamma|y-\bar{y}|+\frac{(1+c)\beta}{a}|y-\bar{y}|\\ &\quad+\beta\eta(1+c)(1-m)|x-\bar{x}|+\beta\eta(1+c)(1-m)|y-\bar{y}|\\ &\leq\left(r+\frac{2r\eta}{k}+\beta\eta(1+c)(1-m)\right)|x-\bar{x}|\\ &\quad+\left(\gamma+\frac{(1+c)\beta}{a}+\beta\eta(1+c)(1-m)\right)|y-\bar{y}|\\ &\leq H\|X-\bar{X}\|, \end{aligned}$$

where

$$\begin{aligned} H=\max\Big\{&r+\frac{2r\eta}{k}+\beta\eta(1+c)(1-m),\\ &\gamma+\frac{(1+c)\beta}{a}+\beta\eta(1+c)(1-m)\Big\}. \end{aligned}$$

Thus, $G(X)$ satisfies the Lipschitz condition. Consequently, the existence and uniqueness of fractional order system (2) follows. □