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Time-varying Hurst–Hölder exponents and the dynamics of (in)efficiency in stock markets

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ABSTRACT

The increasing empirical evidence against the paradigm that stock markets behave efficiently suggests to relax the too restrictive dichotomy between efficient and inefficient markets. Starting from the idea that financial prices evolve in a continuum of equilibria and disequilibria, we use the Hurst–Hölder exponent to quantify the pointwise degree of (in)efficiency and introduce the notion of α-efficiency. We then define and study the properties of two functions which are used to build indicators providing timely information about the market efficiency. We apply our tools to the analysis of four stock indexes representative of U.S., Europe and Asia.

1. Introduction

One of the crucial issues for the asset pricing theories concerns the thorny notion of *market efficiency*. Even before Fama's (1970) influential article, it was widely accepted that securities markets were informationally efficient: at any time t, the price S_t of an individual stock is *expected* to discount all information $\mathscr{F}_t$ accumulated up to time t as a consequence of the quick spread of news, which should ensure that eventual deviations from equilibrium values cannot last for long. Although, "(...) *the expected value is just one of many possible summary measures of a distribution of returns, and market efficiency per se (i.e., the general notion that prices "fully reflect" available information) does not imbue it with any special importance*" [19], formulating the problem in terms of the expected value of discounted payoffs has become a habit in financial literature, also because - as Fama noted - this "*(...) is the unavoidable price one must pay to give the theory of efficient markets empirical content*". In this view, the martingale condition $\mathbb{E}(Y_{t,T}S_T|\mathscr{F}_t) = S_t$ (where Y is the stochastic discount factor and $t < T$), has provided the benchmark for testing the weak form of the *Efficient Market Hypothesis* (EMH), and the idea that investors could not gain abnormal profits has constituted the premise for the random walk models or for their continuous-time counterparts, the models based on Brownian motion.

The impressive body of literature analyzing the random character of stock prices is somewhat controversial, and the results often depend on both the examined markets and the time span, even when the same statistical tools are employed. In [11] for example a synopsis is reported where about 25 studies referred to more than 25 financial markets reach very different conclusions about the presence of memory in financial time series. To make things more controversial, the first empirical evidence of predictability was ascribed to market inefficiency under the assumption of constant expected returns (see e.g. [45]). As an alternative explanation, further studies proposed the time-varying expected returns ([20]), that in their turn can be generated by time-varying risk aversion [14], long-run consumption risk [3], or time-variation in risk-sharing opportunities [33]. As a consequence, within the EMH paradigm, predictability of returns is considered as a symptom of time-varying expected returns [29]. At the same time, outside this paradigm, it is considered as a symptom of memory and, therefore, as a clue of inefficiency. For a discussion of the empirical evidence that in the last years has made the EMH questionable, see the surveys of [34,46,47].

Since the martingale condition affects precisely the predictability and represents the core of the EMH, we will test it by a topological approach which does not require assuming a specific model for the stochastic discount factor (Section 2). The idea is to measure the smoothness of the graph of the price process by means of its Hurst-Hölder pointwise regularity exponent, by knowing that for a Brownian semimartingale – and asymptotically for non Brownian, well-behaved semimartingales – this value equals $\frac{1}{2}$. This is done in Section 3, which recalls the key concepts of the pointwise regularity for deterministic as well as stochastic functions. The relation between the Hurst-Hölder exponent and the martin-

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gale condition is easily understood: an efficient market is incompatible with the ability to forecast future prices, whereas both sufficiently smooth graphs (differentiable, in the limit) and sufficiently mean-reverting graphs allow forecasts, to some extent.

Assuming the price process to be modeled by a *multifractional Brownian motion*, we discuss an estimator of its pointwise regularity (Section 4) and define (in Section 5) two functions that allow to measure – both pointwise and on any compact time set – whether and how much the market deviates from efficiency, once a confidence interval is assigned. We then analyze the behavior of four stock indexes representative of U.S., Europe and Asia: the Dow Jones Industrial Average, the Footsie 100, the Hang Seng, the Nikkei 225 (Section 6). Our main findings can be summarized as follows:

- (a) financial markets generally alternate periods where efficiency predominates to periods where a significant inefficiency can last even for very long time spans, contrarily to what claimed by EMH;
- (b) inefficiency can be split in momentum (when the Hurst–Hölder exponent is larger than $\frac{1}{2}$) and mean-reversion (when the Hurst-Hölder exponent is smaller than $\frac{1}{2}$). We characterize both, by providing a precise and quantitative meaning to the popular saying "*Markets Take The Stairs Up And The Elevator Down*";
- (c) in the long run, momentum and mean-reversion inefficiencies tend to balance themselves and this hides even long-lasting inefficiencies, when asymptotic estimators are used to evaluate the EMH.

Discussion of results and Conclusions close the work.

2. Weak efficiency and martingales

The EMH states that at any time t the price of any stock fully reflects all available information $\mathscr{F}_t$. This implies that prices change only as a reaction to new information or to (predictable or unpredictable) changes in discount factors. Therefore, eventual deviations from the "fair" value cannot last for long because they would be immediately arbitraged away by traders, who would outperform the market on a risk-adjusted basis. This definition emphasizes the role played by the new information, to such an extent that three variants of the hypothesis are considered: *weak*, *semi-strong*, and *strong*. The weak form assumes that $\mathscr{F}_t$ is represented by the sequence of the sole prices of the traded assets; in the semi-strong form, the current price reflects all public information (e.g., news reports, balance sheets, financial statements) and not only the past prices; finally, in the strong form, the current price reflects both public and private information, even hidden "insider" information.

In order to discuss the link between efficient markets and martingale models, let us consider a discrete-time financial model built on a finite probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with the filtration $(\mathscr{F})_{t=\{0,\dots,T\}}$, i.e. an increasing sequence of σ-algebras $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \ldots \subseteq \mathscr{F}_T$, representing the information available up to time t. Henceforth we will use the short notation for the conditional expectation $\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot|\mathscr{F}_t)$. Assume that in the market $d+1$ assets $(S^0, S^1, \cdots, S^d)$ are traded; as usual, S^0 denotes the risk-free asset and S^k $(k = 1, \dots, d)$ the risky assets. Obviously, S^k_t are nonnegative random variables measurable with respect to $\mathscr{F}_t$. Finally, let w be the wealth at time $t = 0$ and $\phi = (\phi^1, \phi^2, \dots, \phi^d) \in \mathbb{R}^d$ be a strategy, where ϕ^k indicates the units from k^{th} asset held in the portfolio. With the above notation, at time $t = 0, 1, \dots, T$ the wealth $W^{(w,\phi)}$ produced by w and strategy ϕ reads as

$$W_t^{(w,\phi)} = \left(w - \sum_{k=1}^{d} \phi^k S_0^k \right) \frac{1}{\beta_t} + \sum_{k=1}^{d} \phi^k S_t^k. \tag{1}$$

where the process $\beta_t = \frac{S_0^0}{S_t^0}$ is called *deflator*. Notice that $W_0^{(w,\phi)} = w$. Eq. (1) can be written in a more insightful way, representing the discounted wealth as the sum of the initial wealth and the discounted portfolio's price variation:

$$\beta_t W_t^{(w,\phi)} = w + \sum_{k=1}^{d} \phi^k \left(\beta_t S_t^k - S_0^k \right). \tag{2}$$

Recalling that the process Y is a stochastic discount factor (SDF) if:

1. $\mathbb{P}(Y_0 = 1, Y_t > 0) = 1$, and
2. $S_0^k = \mathbb{E}_t^{\mathbb{P}}[Y_t S_t^k] \quad (k = 0, \cdots, d)$,

efficiency requires Eq. (1) to comply

$$\mathbb{E}_0^{\mathbb{P}}\left[Y_t W_t^{(w,\phi)} \right] = Y_0 W_0^{(w,\phi)} =: w \tag{3}$$

meaning that, once the EMH is formulated in terms of expected return theories, process $YW^{(w,\phi)}$ is required to be a $\mathbb{P}$-martingale.

Since $\mathbb{E}_0^{\mathbb{P}}\left[Y_T S_T^0 \right] = S_0^0 > 0$, the probability mass

$$Q(\omega) = \frac{S_T^0(\omega)}{S_0^0} \cdot Y_T(\omega) \cdot P(\omega) \tag{4}$$

defines on Ω the risk-neutral probability (or equivalent martingale measure) $\mathbb{Q} \sim \mathbb{P}$, by means of which relation $S_0^k = \mathbb{E}_0^{\mathbb{P}}[Y_T S_T^k]$ turns to be

$$S_0^k = \mathbb{E}_0^{\mathbb{Q}}\left[\frac{S_0^0}{S_T^0} S_T^k \right] = \mathbb{E}_0^{\mathbb{Q}}\left[\beta_T S_T^k \right] \quad (k = 0, \cdots, d), \tag{5}$$

simply stating that the discounted process βS is a $\mathbb{Q}$-martingale.

Dividing equation in condition 2 by S_0^k, after simple manipulations, one obtains the relationship for the returns $R_T^k = \frac{S_T^k}{S_0^k}$, that is in terms of martingale difference sequence

$$\mathbb{E}_0^{\mathbb{P}}\left(Y_T \left(R_T^k - 1 \right) \right) \simeq \mathbb{E}_0^{\mathbb{P}}\left(Y_T \ln R_T^k \right) = 0. \tag{6}$$

Traditionally, most of the empirical literature tests the EMH by checking relation (6) or its variants. Standard approaches have analyzed linear and nonlinear forms of autocorrelation in returns (see e.g. [13,31,44]) or time-varying data (see, e.g., [3,14,19,33]). The main issue in this concern is represented by the so called *joint hypothesis problem*. Indeed, testing (6) implies simultaneously testing both the asset pricing model embedded in the SDF Y_T and the martingale condition itself.

In this paper, we will assume a different perspective, which bypasses the joint hypothesis problem by exploiting some geometrical properties of the paths of a Brownian martingale (and, under mild conditions, even of non Brownian martingales). In particular, we will deduce the martingale behavior by the value of the local regularity Hurst–Hölder exponent of its path. This quantity will be introduced in the next Section.

3. Pointwise regularity

The easiest way to introduce the pointwise regularity of a stochastic process is maybe starting from the Hölder condition, which for a real or complex-valued function f on d-dimensional Euclidean space states, for all x and y in the domain of f and such that $||x - y|| < 1$:

$$|f(x) - f(y)| \leq C||x - y||^{\tilde{\alpha}}$$

where C and $\tilde{\alpha}$ are nonnegative real constants. If:

- $\tilde{\alpha} = 0$, f is bounded;
- $\tilde{\alpha} = 1$, f satisfies a Lipschitz condition;
- $0 < \tilde{\alpha} < 1$, f is continuous and displays fractal properties.