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Tertiary and Quaternary States in the Taylor-Couette System

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ABSTRACT

The analysis of the Taylor-Couette problem in the small gap limit is extended to the cases of tertiary and quaternary solutions. The theoretical results are compared with experimental observations. Although in the latter the small-gap approximation is not always well approximated, the comparison of theoretical results and observations yields reasonable agreements. The absence of the wavy twist mode in the observed patterns is explained by the presence of no-slip boundary conditions in the axial direction of the experimental apparatus, which differ from the periodic conditions imposed in the theoretical analysis. Quaternary solutions bifurcating from the tertiary ones through subharmonic instabilities are presented and compared with experimental observations. Reasonable agreement has been found.

1. Introduction

The Taylor-Couette system continues to fascinate experimental as well as theoretical fluid dynamicists by its rich variety of flow patterns. This variety is best illustrated by the famous Fig. 1 of the paper [1], in which the numerous observed patterns have been indicated as functions of the rotation rates of two independently rotating co-axial cylinders. Since that time several new experimental and theoretical studies have been prepared and this research is likely to continue in the future since a complete understanding of the various patterns of fluid flow in the Taylor-Couette system can still not be claimed. For a recent assessment of the role of the Taylor-Couette problem in the general field of fluid dynamics we refer to [2] and for an earlier review to [3].

In the present paper extensions of the analysis of [4] (to which we shall refer to in the following by WBN) will be presented. Our investigations are motivated by the experimental work of [5] (to which we shall refer to in the following by HBA) and by the more recent plane-Couette flow experiments carried out on a turntable by [6] (referred to hereafter by TTA) and by [7] (referred to by SSA).

In the paper WBN the small-gap approximation was employed for the theoretical analysis of the Taylor-Couette problem. In the small gap limit a symmetry is gained and the number of parameters is reduced by one. The fact that the more recent experimental observations of HBA have confirmed the stability boundaries for the onset of twist vortices even though the radius ratio of their cylindrical boundaries, $r_i/r_o = 0.88$, differed considerably from the small gap limit, $r_i/r_o \approx 1$, encourages us to continue the use of the small gap approximation. The small gap approximation or equivalently the plane Couette flow problem in a system rotating about an axis in the span-wise direction has also been employed by [8] and recently by [9]. [8] has focussed the attention on low rotation rates and referred to [10] for comparisons with experimental observations. Similarly [9] emphasized low rotation rates with the goal of comparisons with the experimental results of TTA and SSA. Because these experiments employed a Couette apparatus on a turn-table the observations had to be restricted to relative small values of the rotation rate. In the present work larger values of the rotation rate will be considered and comparisons will be made with observations of HBA based on a cylindrical apparatus.

A comparison with the observations of HBA has also been performed by [11] (to which we shall refer in the following by AnS) who performed numerical computations of patterns of the Taylor-Couette system for different radius ratios of the cylindrical boundaries. Among those ratios they included the case $r_i/r_o = 0.88$ used by HBA. Wherever a comparison with the results of WBN has been possible AnS found good agreement. We regard this as additional support for our use of the small gap approximation. On the other hand the use of the experimental aspect ratio is not sufficient for a perfect agreement between theory and observation since the usual employment of periodic boundary conditions in the axial direction causes deviations from the experimental conditions.

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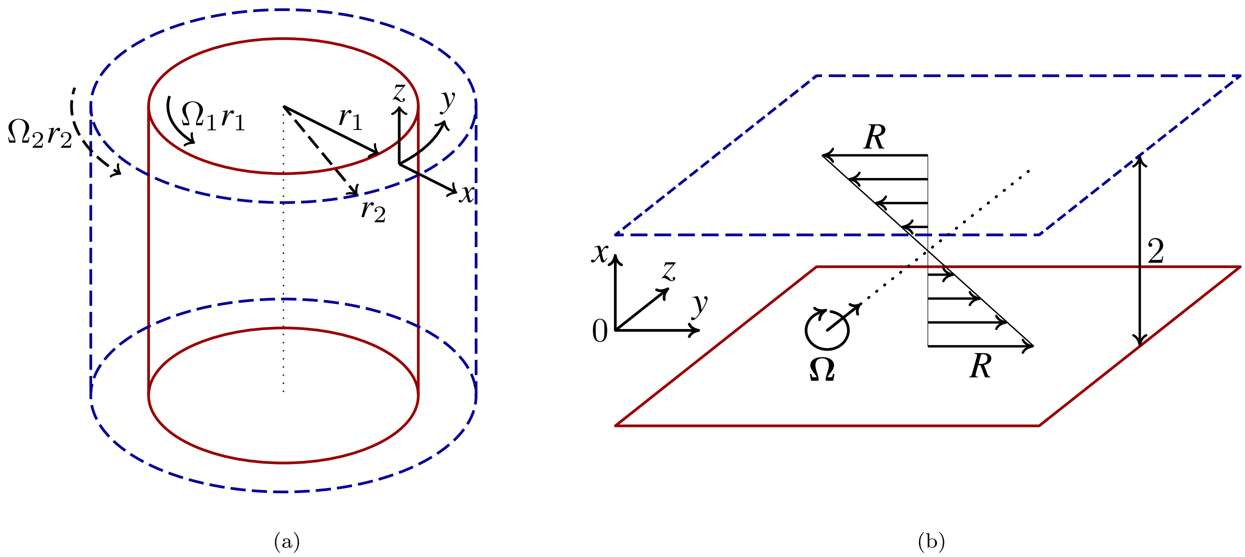

Fig. 1. Taylor-Couette systems. (a) Circular Taylor-Couette system: two co-axial cylinders of radii r_1 and r_2, rotating with different speeds $\Omega_1 r_1$ and $\Omega_2 r_2$, respectively. (b) Narrow gap limit of the circular Taylor-Couette system (rotating plane Couette flow): the system is rotating with angular velocity $\mathbf{\Omega}(=(0,0,\Omega/2))$ and the two plates are drifting with a relative velocity (0, 2R, 0). The dimensional gap width between the two infinite plates is 2h. The x, y and z coordinates in the narrow gap planar system (b) correspond to the radial, azimuthal and axial directions in the circular Taylor-Couette system (a), respectively. Note that (b) refers to the dimensionless system.

The basic equations are formulated and discussed in section 2. There we also describe the numerical methods applied for their solutions. In section 3 the basic properties of the small gap Taylor-Couette system will be surveyed in terms of Taylor vortices with the critical wavelength and their subsequent bifurcations. As has been demonstrated by HBA the properties of solutions bifurcating from axisymmetric Taylor vortices and from tertiary solutions depend rather strongly on the basic wavelength of the vortices. Hence in section 4 we outline stability boundaries for four of the different basic wavelengths investigated by HBA. Our conclusions are given in section 5.

2. Mathematical formulation of the problem

We consider the flow in the narrow gap between two co-axial cylinders with radii r_1 and r_2 that are rotating with speeds $\Omega_1 r_1$ and $\Omega_2 r_2$, respectively. Half of the gap width, $h\equiv(r_2-r_1)/2$ will be used as length scale in the following and h^2/ν is used as timescale, where ν is the kinematic viscosity of the fluid. We assume the limit h/r_1 tending to zero and introduce a Cartesian system of coordinates with x, y, z in the radial, azimuthal and axial directions, respectively, as shown in Fig. 1. The corresponding unit vectors are denoted by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The dimensionless Navier-Stokes equations can then be obtained in the form:

$$\left[\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right]\boldsymbol{u}+\Omega\,\boldsymbol{k}\times\boldsymbol{u}=-\nabla\pi+\nabla^2\boldsymbol{u}, \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{2}$$

where Ω is twice the mean rotation rate in dimensionless units,

$$\Omega=(\Omega_1+\Omega_2)h^2/\nu. \tag{3}$$

The boundary conditions are given by

$$\boldsymbol{u}=\mp R\boldsymbol{j}\quad\text{at}\quad x=\pm1, \tag{4}$$

where the Reynolds number R is defined by

$$R\equiv h(\Omega_1-\Omega_2)(r_1+r_2)/(4\nu). \tag{5}$$

It is convenient to eliminate the equation of continuity (2) by the introduction of the following general representation of the velocity field :

$$\boldsymbol{u}=[-Rx+V(t,x)]\boldsymbol{j}+W(t,x)\boldsymbol{k}+\tilde{\boldsymbol{u}},$$
$$\tilde{\boldsymbol{u}}=\nabla\times[\nabla\times\boldsymbol{i}\phi(t,\boldsymbol{x})]+\nabla\times\boldsymbol{i}\psi(t,\boldsymbol{x}). \tag{6}$$

Accordingly $\tilde{\boldsymbol{u}}$ can be expressed in the form

$$\tilde{\boldsymbol{u}}=-\triangle_2\phi\,\boldsymbol{i}+\left[\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\psi}{\partial z}\right]\boldsymbol{j}+\left[\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial\psi}{\partial y}\right]\boldsymbol{k}, \tag{7}$$

where the operator $\triangle_2$ is defined by $\triangle_2\equiv\nabla^2-(\boldsymbol{i}\cdot\nabla)^2$. By operating with $\boldsymbol{i}\cdot\nabla\times(\nabla\times\circ)$ and $\boldsymbol{i}\cdot\nabla\times\circ$ on Eq. (1) we obtain the following two equations for $\phi(t,\boldsymbol{x})$ and $\psi(t,\boldsymbol{x})$:

$$\begin{aligned}&\left[\nabla^2-\frac{\partial}{\partial t}\right]\nabla^2\triangle_2\phi-\Omega\frac{\partial}{\partial z}\triangle_2\psi\\&=(-Rx+V)\frac{\partial}{\partial y}\nabla^2\triangle_2\phi-\frac{\partial^2 V}{\partial x^2}\frac{\partial}{\partial y}\triangle_2\phi\\&\quad-\frac{\partial^2 W}{\partial x^2}\frac{\partial}{\partial z}\triangle_2\phi+W\frac{\partial}{\partial z}\nabla^2\triangle_2\phi\\&\quad+\boldsymbol{i}\cdot\nabla\times[\nabla\times(\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}})],\end{aligned} \tag{8}$$

$$\begin{aligned}&\left[\nabla^2-\frac{\partial}{\partial t}\right]\triangle_2\psi+\Omega\frac{\partial}{\partial z}\triangle_2\phi\\&=(-Rx+V)\frac{\partial}{\partial y}\triangle_2\psi+\left[R-\frac{\partial V}{\partial x}\right]\frac{\partial}{\partial z}\triangle_2\phi\\&\quad+W\frac{\partial}{\partial z}\triangle_2\psi+\frac{\partial W}{\partial x}\frac{\partial}{\partial y}\triangle_2\phi-\boldsymbol{i}\cdot\nabla\times(\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}}).\end{aligned} \tag{9}$$

The mean flows in the azimuthal and axial directions obey the equations

$$\left[\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right]V=-\frac{\partial}{\partial x}\overline{\triangle_2\phi\left[\frac{\partial^2}{\partial x\partial y}\phi+\frac{\partial}{\partial z}\psi\right]}, \tag{10}$$

$$\left[\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right]W=-\frac{\partial}{\partial x}\overline{\triangle_2\phi\left[\frac{\partial^2}{\partial z\partial x}\phi-\frac{\partial}{\partial y}\psi\right]}, \tag{11}$$

where the bar indicates the average over surfaces $x=$ constant.