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Generalized Tikhonov methods for an inverse source problem of the time-fractional diffusion equation

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ABSTRACT

In this paper, we identify the unknown space-dependent source term in a time-fractional diffusion equation with variable coefficients in a bounded domain where additional data are consider at a fixed time. Using the generalized and revised generalized Tikhonov regularization methods, we construct regularized solutions. Convergence estimates for both methods under an *a-priori* and *a-posteriori* regularization parameter choice rules are given, respectively. Numerical example shows that the proposed methods are effective and stable.

1. Introduction

Fractional diffusion equations have attracted great attention in last few decades. The fractional diffusion equation is a generalization of the classical diffusion equation which models anomalous diffusive phenomena. However, for a few practical issues, the initial information, or a part of boundary information, or diffusion coefficients, or source term might not be given and that we need to recover them by extra measuring information which is able to yield to some fractional diffusion inverse problems. In recent years, inverse problems for time-fractional diffusion equation have become very active, interdisciplinary research area and have wide application in science, engineering, industry, medicine, finance as well as in life and earth sciences.

In this paper, we consider the inverse problem of determining the unknown source term $f(x)$ in time fractional diffusion equation.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficient smooth boundary $\partial\Omega$. We consider the fractional diffusion equation of the form

$$\left.\begin{array}{llll} {}_0\partial_t^\alpha u(x,t) - (Lu)(x,t) = f(x)q(t), & x \in \Omega, & t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, & t \in (0,T), \\ u(x,0) = 0, & x \in \overline{\Omega}, & \end{array}\right\} \quad (1.1)$$

where ${}_0\partial_t^\alpha u$ is the left-sided Caputo fractional derivative of order α $(0<\alpha<1)$ defined by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^\alpha}\frac{\partial u(x,s)}{\partial s}ds, \quad 0<\alpha<1, \quad (1.2)$$

and $-L$ is a symmetric uniformly elliptic operator defined on $D(-L) = H^2(\Omega)\cap H_0^1(\Omega)$ by

$$Lu(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right) + c(x)u(x), \; x \in \Omega, \quad (1.3)$$

the coefficients satisfy $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$, $\sum_{i,j=1}^d a_{ij}\xi_i\xi_j \geq \theta\sum_{i=1}^d |\xi_i|^2 (\theta > 0)$, $c(x) \leq 0, c(x) \in C(\bar{\Omega})$.

The inverse problem is to find $f(x)$ from a final data $u(x,T) = g(x)$. Since the data $g(x)$ is measured, there must be measurement errors and we assume the measured data function $g^\delta(x) \in L^2(\Omega)$

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which satisfies

$$\left\|g-g^{\delta}\right\| \le \delta \tag{1.4}$$

where $\|\cdot\|$ denotes the L^2 norm and the constant $\delta > 0$ represents noise level.

According to the Hadamard requirements (existence, uniqueness and stability of the solution), the inverse problem is ill-posed mathematically. For stable reconstruction, we require some regularization techniques see [1].

If $\alpha = 1$, the problem is inverse problem for standard diffusion equation and has been studied in [2–4]. However, for the fractional inverse source problem, there are only very few works; [5] obtained the uniqueness in determining diffusion coefficient on the basis of Gel'fand-Levitan theory. Sakamoto and Yamamoto [6] derived the regularity and qualitative properties of solution to fractional diffusion-wave equation. Jiang et al. [7] proved uniqueness and Fujishiro and Kian [8] studied stability of the inverse problem of determining a source term with interior measurements in fractional diffusion equation.

When $q(t) = 1$, a number of regularization techniques including Tikhonov regularization and a simplified Tikhonov regularization method [9], quasi-reversibility method [10], truncation method [11] have been applied for the inverse source problem for fractional diffusion equation. According to our knowledge, there are few articles dealing the source term with $q(t)$ such as modified quasi-boundary value method [12], Tikhonov regularization method [13] and the Fourier transform method [14].

The generalized Tikhonov regularization [3] and the revised generalized Tikhonov regularization [4] have been proposed for solving the inverse problems for usual partial differential equations.Yang et al. [15] constructed generalized regularization method for inverse source problem for space-fractional diffusion equation. Zhang and Zhang [16] used generalized Tikhonov method to solve backward time-fractional diffusion problem.

Motivated by above reasons, in this article, we propose generalized and revised generalized Tikhonov regularization methods for inverse source problem for time-fractional diffusion equation with variable coefficients in a general bounded domain. We establish a convergence estimates under an *a-priori* and *a-posteriori* regularization parameter choice rules. All the numerical results are based on the a posteriori parameter choice rule which is independent of the a priori bound of the exact solution. It is more useful in practical issues.

The paper is organized as follows. In Section 2, we simply recall some preliminaries. In Section 3 we construct the regularized solutions by the generalized Tikhonov regularization method and give convergence estimates under an *a-priori* and *a-posteriori* regularization parameter choice rules. In Section 4 we establish the revised generalized Tikhonov regularization method and give convergence estimates under two regularization parameter choice rules. Finally numerical example and its simulation are exploited to demonstrate the usefulness and effectiveness of the methods.

2. Preliminaries

In this section, we recall basic definitions and lemmas.

Definition 2.1. [17,18] The Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.2. *[12, Lemma 2.4] For $\alpha > 0$ and $\beta \in R$, we have:*

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \ z \in C.$$

Lemma 2.3. *[12, Lemma 2.1] Let $\lambda > 0$, then we have:*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0, \quad 0 < \alpha < 1.$$

Lemma 2.4. *[12, Lemma 2.3] For $0 < \alpha < 1$, $\eta > 0$, we have $0 \le E_{\alpha,\alpha}(-\eta) \le \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Lemma 2.5. *[12, Lemma 2.6] For any λ_n satisfying $\lambda_n \ge \lambda_1 > 0$, there exist positive constants $\underline{C}$, depending on α, T, λ_1 such that,*

$$\frac{\underline{C}}{\lambda_n T^{\alpha}} \le E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha}) \le \frac{1}{\lambda_n T^{\alpha}}$$

Lemma 2.6. *[12, Remark 3.1] If $q(t) \in C[0, T]$ satisfying $q(t) \ge q_0 > 0$ for all $t \in [0, T]$, set $\|q\|_{C[0,T]} = \sup_{t\in[0,T]} |q(t)|$. Then we have:*

$$\frac{q_0\underline{C}}{\lambda_n} \le \int_0^T q(\tau)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^{\alpha})d\tau \le \frac{\|q\|_{C[0,T]}}{\lambda_n} \tag{2.1}$$

Proof. By using previous lemmas it is easy to prove (2.1). We omit the details here. □

Lemma 2.7. *For constants $p > 0$, $\mu > 0$, $s \ge \lambda_1 > 0$, we have*

$$F(s) = \frac{s}{q_0\underline{C}(1+\mu s^{p+1})} \le C_1(q_0\underline{C}, p)\mu^{-\frac{1}{p+1}}, \tag{2.2}$$

$$G(s) = \frac{\mu s^{\frac{p}{2}+1}}{1+\mu s^{p+1}} \le C_2(p)\mu^{\frac{p}{2p+2}}, \tag{2.3}$$

$$H(s) = \frac{\|q\|_{C[0,T]}\mu s^{\frac{p}{2}}}{1+\mu s^{p+1}} \le C_3(\|q\|_{C[0,T]}, p)\mu^{\frac{p+2}{2p+2}}. \tag{2.4}$$

Proof. We know that, $\lim_{s\to 0} F(s) = \lim_{s\to\infty} F(s) = 0$, thus $F(s) \le \sup_{s>0} F(s) \le F(s_0)$, where $s_0 > 0$ such that $F'(s_0) = 0$. It is easy to prove that $s_0 = \left(\frac{1}{p\mu}\right)^{1/p+1} > 0$, then we have

$$F(s) \le F(s_0) = \frac{p}{q_0\underline{C}(p+1)}p^{-\frac{1}{p+1}}\mu^{-\frac{1}{p+1}} = C_1(q_0\underline{C}, p)\mu^{-\frac{1}{p+1}}.$$

Similarly, we can prove (2.3) and (2.4). □

3. A generalized Tikhonov regularization method

Denote the eigenvalues of the operator $-L$ as λ_n which satisfy $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and the corresponding eigenfunctions as $X_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ form an orthonormal basis in $L^2(\Omega)$.

As in [6], the fractional power $(-L)^{\gamma}$ is defined for $\gamma \in \mathbb{R}$ and for example $D((-L)^{\frac{1}{2}}) = H_0^1(\Omega)$. We set $\|u\|_{D((-L)^{\gamma})} = \|(-L)^{\gamma}u\|_{L^2(\Omega)}$. We note that the norm $\|u\|_{D((-L)^{\gamma})}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma > 0$.

Define

$$D\big((-L)^{\gamma}\big) = \left\{\psi \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma}|(\psi, X_n)|^2 < \infty\right\},$$

where $(\cdot, \ \cdot)$ is the inner product in $L^2(\Omega)$, then $D((-L)^{\gamma})$ is a Hilbert space with the norm

$$\|\psi\|_{D((-L)^{\gamma})} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma}|(\psi, X_n)|^2\right)^{\frac{1}{2}}.$$