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Residual symmetry, interaction solutions and consistent tanh expansion solvability for the third-order Burgers equation

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ABSTRACT

The residual symmetries for the third-order Burgers equation are obtained with the truncated Painlevé method. The nonlocal symmetries can be localized to the Lie point symmetries by introducing auxiliary dependent variables and the corresponding finite transformations are computed directly. New exact solutions of the third-order Burgers equation is also proved to have the consistent tanh expansion form. New exact interaction excitations such as soliton-cnoidal wave solutions and soliton-periodic wave solutions are given out analytically and graphically.

1. Introduction

As well known, the Burgers hierarchy can be written in the form

$$u_t = K_m(u) = (\partial_x + u + u_x\partial_x^{-1})^{m-1}u_x, \quad m = 1, 2, 3, \cdots, \tag{1}$$

which is of great importance in nonlinear theory and has wide applications in many physical fields [1–7]. In particular, if $m = 3$, then we have the third-order Burgers equation, that is to say,

$$u_t = (\partial_x + u + u_x\partial_x^{-1})^2u_x = 3uu_{xx} + 3u_x^2 + 3u^2u_x + u_{xxx}. \tag{2}$$

The Eq. (2) is also the well-known Sharma–Tasso–Olver equation, which has been studied extensively in many papers [8–12]. For convenience, we change the form of the Eq. (2) by the transformation $t \to -t$, then the third-order Burgers equation or the Sharma–Tasso–Olver equation becomes

$$u_t + 3uu_{xx} + 3u_x^2 + 3u^2u_x + u_{xxx} = 0, \tag{3}$$

which arises in many physical and engineering fields, such as the fluid mechanics, plasma physics and statistical physics and so on. The Painlevé property, recursion operator method and Bäcklund transformation for the third-order Burgers equation are studied in [7]. The authors obtained the fission and fusion of the solitary wave and the soliton solutions of the Eq. (3) by the means of Hirota bilinear method in [12]. The modified simple equation method, the extended tanh expansion method and multi-soliton fusion and fission phenomenon of the Eq. (3) are studied in detail in [13–15].

Recently, abundant interaction solutions among solitons and other complicated waves including periodic cnoidal waves, Painlevé waves and Boussinesq waves for many integrable systems were obtained by nonlocal symmetries reduction and the consistent tanh expansion method related to the Painlevé analysis [16–18]. Hinted at by the results of nonlocal symmetry reduction, Lou found that the symmetry related to the Painlevé truncated expansion is just the residue with respect to the singular manifold in the Painlevé analysis procedure and called residual symmetry [19]. Furthermore, the author proposed a simple effective method, the consistent tanh expansion (CTE) method in [20], which is based on the symmetry reductions with nonlocal symmetries. The CTE method can be used to identify CTE solvable systems and it is a more generalized but much simpler method to look for new interaction solutions between a soliton and other types of nonlinear excitations [21–23].

In this paper, we focus on the residual symmetry and interaction solutions for the third-order Burgers or Sharma–Tasso–Olver Eq. (3). In Section 2, the residual symmetry related to the Painlevé truncated expansion, which is called nonlocal symmetry, is obtained and the corresponding finite transformation is derived by solving the initial value problem of the enlarged system. Section 3 is devoted to the consistent tanh expansion method

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for the third-order Burgers equation and different interaction solutions among different nonlinear excitations such as cnoidal periodic waves and solitary waves are given both analytically and graphically. Last section is summary and discussions.

2. Residual symmetry and its localization

In this section, the standard WTC approach is applied briefly to demonstrate that the third-order Burgers equation (3) can pass the Painlevé test. Let us introduce the Laurent series as

$$u=\sum_{j=0}^{\infty}u_j\phi^{j+\alpha}, \tag{4}$$

with a sufficient number of arbitrary functions among u_j in addition to the arbitrary function ϕ and the negative integer $\alpha=-1$ by the leading order analysis. Substituting the Eq. (4) into the Eq. (3) with $\alpha=-1$, we find the three resonances located at $j=-1,1,3$ corresponding to the arbitrariness of three functions $\{\phi,\ u_1,\ u_3\}$ according to the standard WTC method. Then the third-order Burgers Eq. (3) can be proved to be Painlevé integrable [7] and the truncated Painlevé expansion reads

$$u=\frac{u_0}{\phi}+u_1, \tag{5}$$

Substituting (5) into the Eq. (3) and collecting the coefficients of different powers of ϕ^j, $(j=0,-1,-2,-3,-4)$, we have

$$u_0=\phi_x, \tag{6}$$

$$3\phi_x^2u_{1x}+3u_1^2\phi_x^2+3u_1\phi_{xx}\phi_x+\phi_{xxx}\phi_x+\phi_x\phi_t=0, \tag{7}$$

$$u_{1t}+3u_1u_{1xx}+3u_{1x}^2+3u_1^2u_{1x}+u_{1xxx}=0. \tag{8}$$

It is clear that Eq. (8) is just (3) with the solution u_1 and the residual u_0 is the symmetry corresponding to the solution u_1 based on the residual symmetry theorem in [24]. So the truncated Painlevé expansion

$$u=\frac{\phi_x}{\phi}+u_1, \tag{9}$$

is an auto-Bäcklund transformation between the solutions u and u_1 if the latter is related to ϕ by Eq. (7).

For the nonlocal symmetry (6), the corresponding initial value problem is

$$\frac{d\hat{u}}{d\epsilon}=\hat{\phi}_x,\quad \hat{u}(\epsilon)\mid_{\epsilon=0}=u, \tag{10}$$

with ϵ being an infinitesimal parameter. However, it is very difficult to solve the Eq. (10) for the new functions $\bar{u}(\epsilon)$ due to the intrusion of the function $\bar{\phi}$ and its derivatives. In order to solve this initial value problem, we prolong the third-order Burgers equation such that the nonlocal symmetries become the local symmetries for the prolonged system by introducing another five new dependent variables as the following

$$\phi_x=g,\quad g_x=h,\quad h_x=m,\quad \phi_t=k,\quad k_x=l. \tag{11}$$

Now the nonlocal symmetry (6) for the third-order Burgers Eq. (3) becomes a Lie point symmetry of the prolonged system including (7), (8) and (11). Then the linearized equations of the prolonged system of (7), (8) and (11) are as follows

$$\sigma^g=\sigma^\phi_x,\quad \sigma^h=\sigma^g_x,\quad \sigma^k=\sigma^\phi_t,\quad \sigma^m=\sigma^h_x,\quad \sigma^l=\sigma^k_x, \tag{12}$$

$$\sigma^{u_1}_t+3u_1^2\sigma^{u_1}_x+6u_1\sigma^{u_1}u_{1x}+6u_{1x}\sigma^{u_1}_x+3u_1\sigma^{u_1}_{xx}+3\sigma^{u_1}u_{1xx}+\sigma^{u_1}_{xxx}=0, \tag{13}$$

$$\phi_x\sigma^\phi_{xxx}+\sigma^\phi_x\phi_{xxx}+3\phi_x\sigma^{u_1}\phi_{xx}+3\sigma^\phi_xu_1\phi_{xx}+6\phi_x\sigma^\phi_xu_{1x}+6\phi_xu_1^2\sigma^\phi_x+6u_1\sigma^{u_1}\phi_x^2+3\phi_xu_1\sigma^\phi_{xx}+3\phi_x^2\sigma^{u_1}_x+\sigma^\phi_x\phi_t+\phi_x\sigma^\phi_t=0. \tag{14}$$

One can easily deduce that the solution of (12), (13) and (14) has the form

$$\sigma^\phi=-\phi^2,\ \sigma^{u_1}=g,\ \sigma^g=-2g\phi,\ \sigma^h=-2g^2-2\phi h,\ \sigma^k=-2k\phi, \tag{15}$$

$$\sigma^m=-6gh-2m\phi,\ \sigma^l=-2kg-2l\phi. \tag{16}$$

Correspondingly, the initial value problem becomes

$$\frac{d\hat{u}_1}{d\epsilon}=\hat{g},\quad \frac{d\hat{\phi}}{d\epsilon}=-\hat{\phi}^2,\quad \frac{d\hat{g}}{d\epsilon}=-2\hat{g}\hat{\phi},\quad \frac{d\hat{h}}{d\epsilon}=-2\hat{g}^2-2\hat{h}\hat{\phi},$$
$$\frac{d\hat{k}}{d\epsilon}=-2\hat{k}\hat{\phi},\quad \frac{d\hat{m}}{d\epsilon}=-6\hat{g}\hat{h}-2\hat{m}\hat{\phi},\quad \frac{d\hat{l}}{d\epsilon}=-2\hat{k}\hat{g}-2\hat{l}\hat{\phi},$$
$$\hat{u}_1(\epsilon)\mid_{\epsilon=0}=u_1,\quad \hat{\phi}(\epsilon)\mid_{\epsilon=0}=\phi,\quad \hat{g}(\epsilon)\mid_{\epsilon=0}=g,\quad \hat{h}(\epsilon)\mid_{\epsilon=0}=h,$$
$$\hat{k}(\epsilon)\mid_{\epsilon=0}=k,\quad \hat{m}(\epsilon)\mid_{\epsilon=0}=m,\quad \hat{l}(\epsilon)\mid_{\epsilon=0}=l,$$

then the solutions of the enlarged system (7), (8) and (11) can be written as

$$\hat{\phi}=\frac{\phi}{1+\epsilon\phi},\ \hat{u}_1=u_1+\frac{\epsilon g}{1+\epsilon\phi},\ \hat{g}=\frac{g}{(1+\epsilon\phi)^2},\ \hat{k}=\frac{k}{(1+\epsilon\phi)^2},$$
$$\hat{h}=\frac{h}{(1+\epsilon\phi)^2}-\frac{2\epsilon g^2}{(1+\epsilon\phi)^3},\ \hat{l}=\frac{l}{(1+\epsilon\phi)^2}-\frac{2\epsilon kg}{(1+\epsilon\phi)^3},$$
$$\hat{m}=\frac{m}{(1+\epsilon\phi)^4}+\frac{6\epsilon^2g(g^2-h\phi)+m\phi^2\epsilon^2+2\epsilon m\phi-6\epsilon gh}{(1+\epsilon\phi)^4}, \tag{17}$$

where $\{u_1,\ \phi,\ g,\ h,\ k,\ m,\ l\}$ is a solution of the prolonged system (7), (8) and (11) and ϵ is an infinitesimal parameter. It is interesting to see that the nonlocal symmetry (6) coming from the truncated Painlevé expansion is just the infinitesimal form of the group (17).

3. Consistent tanh expansion solvability and interaction solutions

For a given nonlinear polynomial system

$$P(\mathbf{x},t,u)=0,\quad \mathbf{x}=(x_1,x_2,\ldots,x_n), \tag{18}$$

we aim to find the following possible truncated expansion solution

$$u=\sum_{j=0}^{n}u_j\tanh^j(w), \tag{19}$$

where w is an undetermined function of $(\mathbf{x},t)$, n should be determined from the leading order analysis of the Eq. (18) and all the expansion coefficient functions u_j should be determined by vanishing the coefficients of different powers $\tanh(w)$ after substituting the Eq. (19) into the nonlinear system (18). If the obtained system for $u_j(j=0,1,\cdots,n)$ and w are consistent, or not over-determined, we say that the expansion (19) is a consistent tanh expansion and the exact solutions of the given nonlinear system (18) have the consistent tanh expansion form.

This effective and simple method has been used to find the interaction solutions between solitons and other types of nonlinear waves such as cnoidal periodic waves, Airy waves and so on [21–23]. By the leading order analysis for the third-order Burgers Eq. (3), we can take the following truncated tanh function expansion

$$u=u_0+u_1\tanh(w), \tag{20}$$