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Theory and applications of a more general form for fractional power series expansion

I. Jaradat*, M. Al-Dolat, K. Al-Zoubi, M. Alquran

Department of Mathematics & Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan

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ABSTRACT

The latent potentialities and applications of fractional calculus present a mathematical challenge to establish its theoretical framework. One of these challenges is to have a compact and self-contained fractional power series representation that has a wider application scope and allows studying analytical properties. In this letter, we introduce a new more general form of fractional power series expansion, based on the Caputo sense of fractional derivative, with corresponding convergence property. In order to show the functionality of the proposed expansion, we apply the corresponding iterative fractional power series scheme to solve several fractional (integro-)differential equations.

1. Introduction

Recently, the “fractional derivative” has been utilized in several real-life models to describe the chaotic behavior of a certain phenomenon for a brief span subject to past conditions. Such existing models are including, but not limited to, viscoelasticity [1–3], quantum mechanics [4–6], electromagnetism [7], electrochemistry [8], signal and image processing [9], vibration and oscillation [10,11], and biology [12,13]. As a result, it increasingly becomes important to construct a mathematical framework for this concept and to solve the associated differential equations. For more details and applications about fractional derivative, we refer the reader to [14–16].

Hitherto several fractional power series expansions have been presented in the literature [17–19]. However, all of them lack sufficient integer exponents for the variable under consideration and/or linear exponents in term of the fractional order derivative $\alpha > 0$. Therefore, our motivation is to provide a more integrated representation of fractional power series with a related convergence theorem. Consequently, in analogy to the classical power series method, we exploit this expansion to obtain closed-form solutions to various types of fractional (integro-)differential equations.

To the best knowledge of the authors, there is no unanimous definition for the term “fractional derivative” since all the proposed approaches do not preserve the classical integer-order derivative properties. However, in the Caputo sense, the derivative of a constant function is zero and the fractional differential equations need not have fractional order initial conditions. For these reasons, we adopt the Caputo fractional derivative approach in our work, which is defined for a function $u \in \mathcal{C}^n(0, \infty)$ by

$$\mathcal{D}_x^{\alpha}[u(x)] = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}}\, d\tau \tag{1.1}$$

when $n-1 < \alpha < n$ and by $\mathcal{D}_x^{\alpha}[u(x)] = u^{(n)}(x)$ when $\alpha = n \in \mathbb{N}$.

It should be noted here that it suffices to consider the Caputo fractional derivative of order $0 < \alpha \leq 1$ since $D_t^{\alpha}[u(t)] = D_t^{\alpha-(n-1)}[u^{(n-1)}(t)]$ for arbitrary order $n-1 < \alpha \leq n$, where $\alpha - (n-1) \in (0, 1]$.

2. Self-contained fractional power series expansion

In this section, we exhibit a coherent representation of fractional power series with a related convergence theorem. Unlike the well-known expansions, the exponents of the indeterminate consist of sufficient positive integers and linear description in term of the fractional derivative order $\alpha > 0$. Throughout the rest of this section, we assume $\alpha \in (0, 1]$.

* Corresponding author.
E-mail addresses: iajaradat@just.edu.jo (I. Jaradat), mmaldolat@just.edu.jo (M. Al-Dolat), kfzoubi@just.edu.jo (K. Al-Zoubi), marwan04@just.edu.jo (M. Alquran).

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Definition 2.1. A generalized fractional power series (GFPS) centered at $x_0 = 0$ is an infinite series of the form

$$\sum_{i+j=0}^{\infty} c_{ij} x^{i\alpha+j} = \underbrace{c_{00}}_{i+j=0} + \underbrace{c_{01}x^1 + c_{10}x^{\alpha}}_{i+j=1} + \underbrace{c_{02}x^2 + c_{11}x^{\alpha+1} + c_{20}x^{2\alpha}}_{i+j=2} + \dots \tag{2.1}$$

where $i, j \in \mathbb{N}^*$, $x \geq 0$ is a variable of indeterminate, and c_{ij}'s are the coefficients of the series.

Conveniently, we here assumed that the center of GFPS (2.1) is zero since this can always be done via the linear change of variable $(x - x_0) \mapsto x$.

We remark here that the GFPS expansion (2.1) generalizes all the well-known comparable expansions in the literature [17–19] in the sense that it includes more integer exponents and linear representations of α. In particular when $\alpha = 1$, we have the classical power series expansion $\sum_{k=0}^{\infty} c_k x^k$ where $c_k = \sum_{i+j=k} c_{ij}$. Moreover, the GFPS (2.1) is naturally obtained as a Cauchy product of two power series, after rearrangement, as following

$$\sum_{i+j=0}^{\infty} c_{ij} x^{i\alpha+j} = \left(\sum_{i=0}^{\infty} a_i x^{i\alpha}\right)\left(\sum_{j=0}^{\infty} b_j x^{j}\right) \tag{2.2}$$

where $c_{ij} = a_i b_j$.

Remark 1. A direct implementation of the Caputo fractional derivative yields

$$\mathcal{D}_x^{\alpha}[x^{\beta}] = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta > 0 \\ 0, & \beta = 0. \end{cases} \tag{2.3}$$

Thus, by term-by-term differentiation within the interval of convergence of $x > 0$, if $u(x) = \sum_{i+j=0}^{\infty} c_{ij} x^{i\alpha+j}$, then

$$\mathcal{D}_x^{\alpha}[u(x)] = \sum_{i+j=0}^{\infty} \frac{\Gamma((i+1)\alpha+j+1)}{\Gamma(i\alpha+j+1)} c_{i+1,j} x^{i\alpha+j} + \sum_{j=1}^{\infty} \frac{j!}{\Gamma(j+1-\alpha)} c_{0j} x^{j-\alpha}, \tag{2.4}$$

and

$$\mathcal{D}_x^{2\alpha}[u(x)] = \sum_{i+j=0}^{\infty} \frac{\Gamma((i+2)\alpha+j+1)}{\Gamma(i\alpha+j+1)} c_{i+2,j} x^{i\alpha+j} + \sum_{j=1}^{\infty} \frac{\Gamma(\alpha+j+1)}{\Gamma(j+1-\alpha)} c_{1j} x^{j-\alpha} + \sum_{j=1}^{\infty} \frac{j!}{\Gamma(j+1-2\alpha)} c_{0j} x^{j-2\alpha} \tag{2.5}$$

where $\mathcal{D}^{2\alpha} = \mathcal{D}^{\alpha}\mathcal{D}^{\alpha}$. And so on.

Proposition 2.2. *If $\sum_{k=0}^{\infty} a_k x^{k\alpha}$ converges for some $x = a > 0$, then it converges absolutely for $x \in (0, a)$.*

Proof. Fix $\epsilon = 1$. Then by the convergence of $\sum_{k=0}^{\infty} a_k a^{k\alpha}$, there is $N \in \mathbb{N}$ such that $|a_k a^{k\alpha}| < 1$ for all $k \geq N$. Thus for $k \geq N$ and $x \in (0, a)$, we have $|a_k x^{k\alpha}| < (\frac{x}{a})^{\alpha k}$ which yields that $\sum_{k=0}^{\infty} |a_k x^{k\alpha}|$ is convergent by comparison test. □

Corollary 2.3. *If $\sum_{k=0}^{\infty} b_k x^k$ converges for some $x = b > 0$, then it converges absolutely for $x \in (0, b)$.*

The next theorem, which is an analog to Mertens' Theorem, gives a necessary condition to guarantee the convergence of (2.1).

Theorem 2.4. *Consider the two power series $A = \sum_{k=0}^{\infty} a_k x^{k\alpha}$ and $B = \sum_{k=0}^{\infty} b_k x^k$ such that A converges absolutely to a for $x = x_a > 0$ and B converges to b for $x = x_b > 0$. Then the Cauchy product of A and B converges to ab for $x = x_c > 0$ where $x_c = min\{x_a, x_b\}$.*

Proof. Let $C = \sum c_k$ denotes the Cauchy product of A and B, and let A_n, B_n, and C_n denote the partial sums of A, B, and C respectively. After rearrangement of terms, we have

$$\begin{aligned} C_n &= \sum_{k=0}^{n} c_k \\ &= \sum_{k=0}^{n}\sum_{i=0}^{k} (a_i x^{i\alpha})(b_{k-i} x^{k-i}) \\ &= \sum_{k=0}^{n} (a_k x^{k\alpha}) B_{n-k} \\ &= b \sum_{k=0}^{n} a_k x^{k\alpha} + \sum_{k=0}^{n} a_k x^{k\alpha}(B_{n-k} - b) \\ &= bA_n + \sum_{k=0}^{n} a_k \widehat{B}_{n-k} x^{k\alpha}, \end{aligned} \tag{2.6}$$

where $\widehat{B}_{n-k} = B_{n-k} - b$. Since $bA_n \xrightarrow{n\to\infty} ab$, then it suffices to show that $\sum_{k=0}^{n} a_k \widehat{B}_{n-k} x^{k\alpha} \xrightarrow{n\to\infty} 0$. To do so, let $\epsilon > 0$. Then, since $\widehat{B}_n \xrightarrow{n\to\infty} 0$, there exists $M > 0$ such that $|\widehat{B}_n| \leq M$ for all $n \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $|\widehat{B}_k| < \frac{\epsilon}{2a}$ for all $k \geq n_0$. Therefore, for $n \geq n_0$ we have

$$\begin{aligned} \left|\sum_{k=0}^{n} a_k \widehat{B}_{n-k} x^{k\alpha}\right| &\leq \sum_{k=0}^{n} \left|a_{n-k} \widehat{B}_k x^{(n-k)\alpha}\right| \\ &= \sum_{k=0}^{n_0} \left|a_{n-k} \widehat{B}_k x^{(n-k)\alpha}\right| + \sum_{k=n_0+1}^{n} \left|a_{n-k} \widehat{B}_k x^{(n-k)\alpha}\right| \\ &\leq M \sum_{k=0}^{n_0} \left|a_{n-k} x^{(n-k)\alpha}\right| + \frac{\epsilon}{2a} \sum_{k=n_0+1}^{n} \left|a_{n-k} x^{(n-k)\alpha}\right| \\ &< M \sum_{k=n-n_0}^{n} \left|a_k x^{k\alpha}\right| + \frac{\epsilon}{2}. \end{aligned} \tag{2.7}$$

Now, by the absolutely convergence of A and the Cauchy criterion, there exits $n_1 \in \mathbb{N}$ such that for all $n > m > n_1$ we have $\sum_{k=m+1}^{n} |a_k x^{k\alpha}| < \frac{\epsilon}{2M}$. Thus, $\sum_{k=n-n_0}^{n} |a_k x^{k\alpha}| < \frac{\epsilon}{2M}$ for all $n > n_0 + n_1$, and hence $|\sum_{k=0}^{n} a_k \widehat{B}_{n-k} x^{k\alpha}| < \epsilon$ for all $n \geq n_0 + n_1$ as desired. □

It should be noted here that we can argue analogously if we swap the convergence rules between A and B in the last theorem.

3. A direct application of GFPS

In this section, the proposed GFPS expansion (2.1) will be utilized to introduce a parallel scheme to the power series solution method to handle various types of fractional (integro-)differential equations. The first two examples are somewhat artificial to show that our proposed expansion is more comprehensive in the sense it can capture more integer and fractional exponents of the indeterminate. Whereas the rest examples are well-known from literature. It should be noted here that all the necessary calculations and graphical analysis are done by using Mathematica 10.

Example 1. Consider the following fractional initial value problem:

$$\mathcal{D}_x^{\alpha}[y(x)] - \lambda y(x) = \frac{x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad y(0) = y_0, \tag{3.1}$$

where $0 < \alpha \leq 1$ and $x \geq 0$. In the light of the previous discussion and using the initial condition, the proposed generalized fractional