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Review

Dynamics towards the steady state applied for the Smith-Slatkin mapping

Juliano A. de Oliveira[a,b,*], Larissa C. N. Ramos[b], Edson D. Leonel[a]

[a] Universidade Estadual Paulista (UNESP), Instituto de Geociências e Ciências Exatas, Departamento de Física, Câmpus de Rio Claro, Av.24A, 1515, SP 13506-900, Brazil

[b] Universidade Estadual Paulista (UNESP), Câmpus de São João da Boa Vista, Av. Profa., Isette Corrêa Fontão, 505, SP 13876-750, Brazil

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ABSTRACT

We derived explicit forms for the convergence to the steady state for a 1-D Smith–Slatkin mapping at and near at bifurcations. We used a phenomenological description with a set of scaling hypothesis leading to a homogeneous function giving a scaling law. The procedure is supported by numerical simulations and confirmed by a theoretical description. At the bifurcation we used an approximation transforming the difference equation into a differential one whose solution remount all scaling features. Near the bifurcation an investigation of fixed point stability leads to the decay for the stationary state. Simulations are made in the pitchfork, transcritical and period doubling bifurcations.

1. Introduction

The pioneer application of nonlinear mapping for the investigation of population dynamics in biology is due to May [1]. After his publication many different contributions appeared. Applications of mappings are vast and can be seen in physics [2–6], chemistry, biology, engineering, mathematics and many others [7–17].

The investigation of stability of fixed points as well as conditions leading to bifurcations are well known [18–21]. Intermittence was investigated in Ref. [22] and led to interesting properties where a pseudo regularity along a chaotic dynamics is anticipating a tangent bifurcation giving birth to a periodic window, hence to regularity. It is known that the convergence to the fixed point at the bifurcation was proved to obey an homogeneous function characterized by a set of three critical exponents [23,24]. Near the bifurcations the dynamics converges to the steady state by means of an exponential decay [23] whose relaxation time is given by a power law for a bifurcation parameter. The set of critical exponents dictates the speed of convergence to the stationary point and can also be used to identify, whenever it is not possible analytically what type the bifurcation is. In this paper, we consider the Smith–Slatkin mapping, derived from applications in biology, and seek to obtain, understand and describe the critical exponents near the bifurcations. We implement different procedures to describe the dynamics and hence obtain the exponents. First we identify where the bifurcations are. Then we investigate the convergence to the fixed point using numerical simulations. We consider and approximation that transforms the difference equation, near the fixed point, into a differential equation, and solve it analytically to compare the arguments with the corresponding scaling times. The critical exponents emerge naturally from such a procedure and are obtained for short and large times. Near the bifurcation we obtain the relaxation time to the steady state by using fixed point stability analysis. At the bifurcation the convergence is described by an homogeneous function while near the bifurcations an exponential decay explains how the steady state is reached.

The organization of the paper is simple. Section 2 describes the mapping, the numerical simulations as well as the analytical findings. Conclusions are shown in Section 3.

2. The model and scaling properties

The model we consider in this paper is a version of the Smith–Slatkin mapping, which is written as [25–28]

$$x_{n+1} = f(x_n) = \frac{R x_n}{1 + a\, x_n^{\gamma}}\ , \tag{1}$$

where R, a and γ are control parameters and we consider them to be non negative. The dynamical variable is represented by x when the index n denotes the iteration number. For the case of $\gamma = 1$ the Skellam model [28,29] is recovered. To give a glimpse of the orbit diagram, Fig. 1 was constructed for the parameters $\gamma = 6$, $a = 1$ for the initial condition $x_0 = 0.01$.

* Corresponding author.
E-mail address: julianoantonio@sjbv.unesp.br (J.A. de Oliveira).

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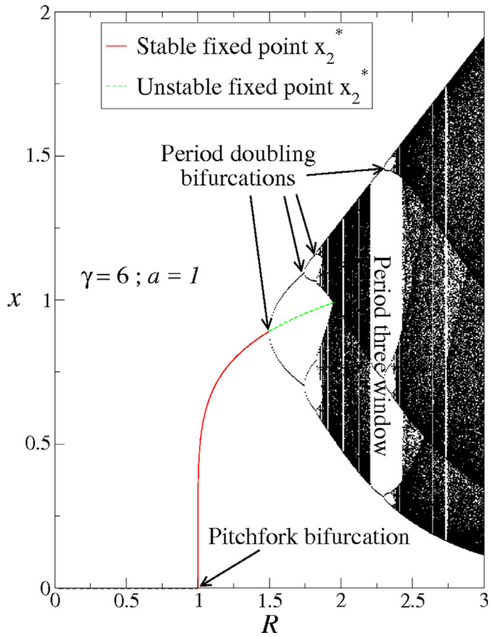

Fig. 1. Orbit diagram obtained for Eq. (1) using $\gamma = 6$, $a = 1$ and the initial condition $x_0 = 0.01$. Fixed point x_2^* is represented in red (stable) and green (unstable). Bifurcations as well as the main periodic window are identified in the diagram. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The fixed points, obtained from the condition $x_{n+1} = x_n = x^*$ are: (i) $x_1^* = 0$, which is asymptotically stable for $R \in [0, 1)$, (ii) and (iii) deserve a short discussion first. For any odd γ, fixed points are: (ii) $x_{2,3}^* = \pm\left[\frac{R-1}{a}\right]^{\frac{1}{\gamma}}$ while γ is of any other kind, even, irrational etc, we obtain (iii) $x_2^* = \left[\frac{R-1}{a}\right]^{\frac{1}{\gamma}}$. Fixed points $x_{2,3}^*$ are asymptotically stable[1] for $R \in (1, \frac{\gamma}{(\gamma-2)})$, for $\gamma \neq 2$. A pitchfork supercritical bifurcation happens at $R = 1$ for an odd γ while a transcritical is observed at same R for any other value of γ. Red curve in Fig. 1 shows the stable fixed point x_2^* while green curve is a continuation of x_2^* however after a period doubling bifurcation where it is unstable. A period two orbit arises at $R = \frac{\gamma}{(\gamma-2)}$ following normal Feigenbaum scaling [30,31] after that.

Our first objective in this paper is to consider the convergence to the fixed point $x_1^* = 0$ at the bifurcation in $R = 1$. We shall show it obeys scaling properties leading different curves generated by different initial conditions to overlap onto each other, after appropriate scaling transformations, into a single and universal plot. Near the bifurcation, the convergence is so far described by an exponential decay whose relaxation time depend on the distance of the bifurcation. We use Taylor expansion near the fixed point, investigating the fixed point stability to prove it.

To illustrate how the dynamical variable evolves to the equilibrium at a bifurcation, we considered $R = 1$, $\gamma = 6$, $a = 1$ and different initial conditions for x_0. Fig. 2(a) shows the convergence to the fixed point $x_1^* = 0$. We see that for short n, the orbit stays confined in a regime of seemingly constant plateau. After a while eventually it suffers a changeover at a typical crossover iteration number denoted as n_x and ultimately bends towards a regime of decay to its final state x^*.

The scaling properties extracted from Fig. 2(a) are the following: (i) For a short $n \ll n_x$ we notice $x(n) \propto x_0^{\alpha}$, leading us to conclude that $\alpha = 1$ since $x(n) \propto x_0$; (ii) For large enough n, typically $n \gg n_x$, the dynamical variable is described as $x(n) \propto n^{\beta}$ where β is a decay exponent which depends on the nonlinearity of the mapping γ. For $\gamma = 6$ we obtained from fitting numerically the data an exponent $\beta = -0.16666320(8)$, as shown in the decaying regime of Fig. 2; (iii) Finally, the crossover iteration number n_x is given by $n_x \propto x_0^{z}$ where z is a changeover exponent.

A homogeneous function of the type

$$x(x_0, n) = l x(l^{\tilde{a}} x_0, l^{\tilde{b}} n) \ , \tag{2}$$

is a natural consequence of the behavior observed from Fig. 2(a) as well as from the scaling hypotheses. Here l is a scaling factor, $\tilde{a}$ and $\tilde{b}$ are characteristic exponents. Doing a similar procedure as made in Ref.[23] a scaling law appears as

$$z = \frac{\alpha}{\beta}. \tag{3}$$

The knowledge of any two exponents allows one to find the third by using Eq. (3). The relevant scaling transformations to be made are $x \to x/x_0^{\alpha}$ and $n \to n/x_0^{z}$, leading to a perfect overlap of all curves shown in Fig. 2(a) onto a single and hence universal curve, as shown in Fig. 2(b).

When the dynamical variable $x(n)$ is very close to the equilibrium, the expression $x_{n+1} = R x_n (1 + a x_n^{\gamma})^{-1}$ can be Taylor expanded leading to $x_{n+1} = R x_n (1 - a x_n^{\gamma})$. Moreover its variation as compared to the next iterate to be very small, i. e., $x_{n+1} - x_n$ is small enough. Such property allows us to use the following approximation $x_{n+1} - x_n \cong \frac{df}{dn}$. For $R = 1$, this leads to $\frac{df}{dn} = -a x^{\gamma+1}$. This is a first order differential equations that must be solved for the ranges $x \in [x_0, x(n)]$ and n starting from $n = 0$. The solution is written as

$$x(n) = \frac{x_0}{[1 + a\gamma x_0^{\gamma} n]^{\frac{1}{\gamma}}}. \tag{4}$$

Eq. (4) allows us to do the following analysis: (i) when $a\gamma x_0^{\gamma} n \ll 1$, we have $x(n) \cong x_0$, therefore leading to $\alpha = 1$; (ii) For the case of $a\gamma x_0^{\gamma} n \gg 1$ we end up with $x(n) \approx n^{-1/\gamma}$, hence $\beta = -1/\gamma$; (iii) for the case $a\gamma x_0^{\gamma} n_x = 1$ we have $n_x \propto x_0^{-\gamma}$, therefore $z = -\gamma$. All of these findings are giving support for the numerical simulations. Eq. (4) is plotted in Fig. 2(a) as dashed lines and we see the agreement between the numerical an analytical description is remarkable.

Near the bifurcation the dynamics is not described anymore by an homogeneous function. Instead of it the convergence is described rather by an exponential decay of the type (see Refs. [32,33])

$$x(n) - x^* = (x_0 - x^*) e^{-n/\tau}, \tag{5}$$

where τ is the relaxation time described by

$$\tau \propto \mu^{\delta}, \tag{6}$$

and δ is a relaxation exponent. Fig. 3 shows the behavior of τ *vs.* μ given an exponent $\delta = -0.9879(4) \cong -1$, obtained by a numerical fitting of the data, and this result is invariant with respect to the parameter γ.

Let us now describe the convergence to the steady state when $R \neq 1$, therefore near the bifurcation. There is no difference on the procedure considering before or after the bifurcation. We shall consider the neighborhood of $R = R_c = 1$, where the index c denotes the critical, i.e., the bifurcation parameter. Starting from an initial condition near the fixed point we have $x_0 = x^* + \epsilon_0$, where x^* denotes the fixed point and ϵ_0 corresponds to an initial distance from the fixed point. Since the mapping is given by $x_{n+1} = f(x_n)$, we have that $x_1 = f(x^* + \epsilon_0)$. Since ϵ_0 is sufficiently small, a Taylor

[1] By asymptotically stable we mean that given an initial condition inside of the basin of attraction of the fixed point, in the limit of $\lim n \to \infty$, the orbit has a final state at x^*, hence converging to the fixed point.