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Counterexamples on Jumarie's three basic fractional calculus formulae for non-differentiable continuous functions

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ABSTRACT

Jumarie proposed a modified Riemann–Liouville derivative definition and gave three so-called basic fractional calculus formulae such as Leibniz rule $(u(t)v(t))^{(\alpha)} = u^{(\alpha)}(t)v(t) + u(t)v^{(\alpha)}(t)$, where u and v are required to be non-differentiable and continuous at the point t. We once gave the counterexamples to show that Jumarie's formulae are not true for differentiable functions. In the paper, we give further counterexamples to prove that in non-differentiable cases these Jumarie's formulae are also not true. Therefore, we proved that Jumarie's formulae are not true for both cases of differentiable and non-differentiable functions, and then those results on fractional soliton equations obtained by using Jumarie's formulae are not right.

1. Introduction

Jumarie proposed a modified Riemann–Liouville fractional derivative [1–5]:

$$f^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-x)^{-\alpha}(f(x)-f(0))\mathrm{d}x, \tag{1}$$

and gave some basic fractional calculus formulae (see, for example, formulae (3.11)-(3.13) in [4] or formulae (4.3), (4.4) and (4.5) in [5]):

$$(u(t)v(t))^{(\alpha)} = u^{(\alpha)}(t)v(t) + u(t)v^{(\alpha)}(t), \tag{2}$$

$$(f(u(t)))^{(\alpha)} = f'_u u^{(\alpha)}(t), \tag{3}$$

where Jumarie requires the functions u and v are non-differentiable and continuous, while f is differentiable at the point t. Jumarie's third formula is given by

$$(f(u(t)))^{(\alpha)} = (f(u))^{(\alpha)}(u'(t))^{\alpha}, \tag{4}$$

where f is non-differentiable and u is differentiable at the point t.

The formula (3) has been applied to solve the exact solutions to some nonlinear fractional order soliton equations(see, for example, [6–9]).

In [10], I once gave three counterexamples to show that Jumarie's these so-called basic formulae are not correct in the case of differentiable functions. In [11], Jumarie emphasizes that it is just at some point that his formulae do hold. At such point, the function is continuous and non-differentiable, and the fractional derivative exists. In the present paper, I provide further counterexamples which satisfy all conditions in Jumarie's formulae to show directly that Jumarie's formulae are incorrect in the case of non-differentiable continuous functions. Finally, I prove that essentially non-differentiable cases can be transformed to the differentiable cases.

Recently, some problems about the rules of fractional derivatives have been discussed by some authors (see, for example, [10–14]). For instance, Tarasov [12,13] gave some important results for Leibniz rule and chain rule. For local fractional derivatives of nowhere differentiable continuous functions on open intervals, some detailed discussions can be found in [14]. Further discussions on some subtle problems of fractional calculus can be found in [15,16].

Remark: Although only one counterexample is enough, I give yet more counterexamples. My purpose to do so is to offer more points of view to understand the problem.

2. Counterexamples to formula (2)

As in [10], we need the $\frac{1}{2}$–order derivatives of the following four functions $f(t) = t$, $f(t) = \sqrt{t}$, $f(t) = t^2$ and $f(t) = t^{\frac{3}{2}}$ with $f(0) = 0$:

$$(t)^{(1/2)} = 2\sqrt{\frac{t}{\pi}}, \tag{5}$$

$$(\sqrt{t})^{(1/2)} = \frac{\sqrt{\pi}}{2}, \tag{6}$$

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$$(t^2)^{(1/2)} = \frac{8t^{3/2}}{3\sqrt{\pi}}, \tag{7}$$

$$(t^{\frac{3}{2}})^{(1/2)} = \frac{3\sqrt{\pi}t}{4}. \tag{8}$$

Counterexample 1 (The counterexample of formula (2)). Take $\alpha = \frac{1}{2}$ and

$$u(t) = \begin{cases} \sqrt{t}, & 0 \le t \le 1, \\ \sqrt{t} + t - 1, & t > 1. \end{cases} \tag{9}$$

It is easy to see that $u(t)$ is continuous, and is non-differentiable at $t = 1$. Further, we have

$$H(t) = \int_0^t (t-x)^{-\alpha}(u(x) - u(0))\mathrm{d}x = \begin{cases} \int_0^t \frac{\sqrt{x}}{\sqrt{t-x}}\mathrm{d}x, & 0 \le t \le 1, \\ \int_0^1 \frac{\sqrt{x}}{\sqrt{t-x}}\mathrm{d}x + \int_1^t \frac{\sqrt{x}+x-1}{\sqrt{t-x}}\mathrm{d}x, & t > 1. \end{cases} \tag{10}$$

And then, we have

$$H(t) = \begin{cases} \int_0^t \frac{\sqrt{x}}{\sqrt{t-x}}\mathrm{d}x, & 0 \le t \le 1, \\ \int_0^t \frac{\sqrt{x}}{\sqrt{t-x}}\mathrm{d}x + \int_1^t \frac{x-1}{\sqrt{t-x}}\mathrm{d}x, & t > 1. \end{cases} \tag{11}$$

By letting $t - x = s^2$, we get

$$K(t) = \int_1^t \frac{x-1}{\sqrt{t-x}}\mathrm{d}x = 2\int_0^{\sqrt{t-1}} (t-1-s^2)\mathrm{d}s = \frac{4}{3}(t-1)^{\frac{3}{2}}. \tag{12}$$

Therefore, if $0 \le t < 1$,

$$u^{(1/2)}(t) = (\sqrt{t})^{(\frac{1}{2})} = \frac{\sqrt{\pi}}{2}, \tag{13}$$

and if $t > 1$,

$$u^{(1/2)}(t) = (\sqrt{t})^{(\frac{1}{2})} + \frac{1}{\sqrt{\pi}}K'(t) = \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\pi}}(t-1)^{\frac{1}{2}}, \tag{14}$$

where we use $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. And hence, at $t = 1$, it follows that $u^{(1/2)}(1)$ exists and

$$u^{(1/2)}(1) = \frac{\sqrt{\pi}}{2}. \tag{15}$$

Further, by taking $v(t) = u(t)$, we get

$$u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1) = \sqrt{\pi}. \tag{16}$$

On the other hand, we have

$$u(t)v(t) = \begin{cases} t, & 0 \le t \le 1, \\ (\sqrt{t} + t - 1)^2, & t > 1. \end{cases} \tag{17}$$

Hence, if $t < 1$, we have

$$(uv)^{(1/2)}(t) = (t)^{(\frac{1}{2})} = 2\sqrt{\frac{t}{\pi}}, \tag{18}$$

and if $t > 1$,

$$(uv)^{(1/2)}(t) = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^1 \frac{x}{\sqrt{t-x}}\mathrm{d}x + \int_1^t \frac{(\sqrt{x}+x-1)^2}{\sqrt{t-x}}\mathrm{d}x\right\}. \tag{19}$$

Further, we have

$$(uv)^{(1/2)}(t) = (t)^{(\frac{1}{2})} + \frac{1}{\sqrt{\pi}}\int_1^t \frac{3\sqrt{x} + 2(x-1) - x^{-\frac{1}{2}}}{\sqrt{t-x}}\mathrm{d}x. \tag{20}$$

By computing the last integral, we get

$$(uv)^{(1/2)}(t) = 2\sqrt{\frac{t}{\pi}} + \frac{1}{\sqrt{\pi}}\left\{\frac{8}{3}(t-1)^{\frac{3}{2}} + 3\sqrt{t-1} + 3t\left(\frac{\pi}{2} - \arcsin\frac{1}{\sqrt{t}}\right) + 2\arcsin\frac{1}{\sqrt{t}} - \pi\right\}. \tag{21}$$

Therefore, at $t = 1$, $(uv)^{(1/2)}(t)$ does exist and $(uv)^{(1/2)}(1) = \frac{2}{\sqrt{\pi}} \ne \sqrt{\pi}$. From (16), it turns out that

$$(uv)^{(1/2)}(1) \ne u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1). \tag{22}$$

This example shows that Jumarie's formula (2) is not true for the non-differentiable continuous functions.

Next, we give a more simple example.

Counterexample 2. Take $\alpha = \frac{1}{2}$ and

$$u(t) = \begin{cases} 1 - t, & t \le 1, \\ t - 1, & t > 1. \end{cases} \tag{23}$$

It is easy to see that $u(t)$ is continuous, and is non-differentiable at $t = 1$. Further, we have

$$H(t) = \int_0^t (t-x)^{-\alpha}(u(x) - u(0))\mathrm{d}x = \begin{cases} \int_0^t \frac{-x}{\sqrt{t-x}}\mathrm{d}x, & t \le 1, \\ \int_0^1 \frac{-x}{\sqrt{t-x}}\mathrm{d}x + \int_1^t \frac{x-2}{\sqrt{t-x}}\mathrm{d}x, & t > 1. \end{cases} \tag{24}$$

And then, we have

$$H(t) = \begin{cases} \int_0^t \frac{-x}{\sqrt{t-x}}\mathrm{d}x, & t \le 1, \\ \int_0^t \frac{-x}{\sqrt{t-x}}\mathrm{d}x + 2\int_1^t \frac{x-1}{\sqrt{t-x}}\mathrm{d}x, & t > 1. \end{cases} \tag{25}$$

Therefore, if $t < 1$,

$$u^{(1/2)}(t) = -(t)^{(\frac{1}{2})} = -2\sqrt{\frac{t}{\pi}}, \tag{26}$$

and if $t > 1$,

$$u^{(1/2)}(t) = \frac{1}{\sqrt{\pi}}H'(t) = -2\sqrt{\frac{t}{\pi}} + 4\sqrt{\frac{t-1}{\pi}}. \tag{27}$$

It follows that

$$u^{(1/2)}(1) = -\frac{2}{\sqrt{\pi}}. \tag{28}$$

Hence, from $u(1) = 0$ we have

$$2u(1)u^{(1/2)}(1) = 0. \tag{29}$$

On the other hand, we have $u^2(t) = (t-1)^2 = t^2 - 2t + 1$, and then

$$(u^2)^{(1/2)}(t) = (t^2)^{(1/2)} - 2(t)^{(1/2)} = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{4}{\sqrt{\pi}}t^{\frac{1}{2}}. \tag{30}$$

Therefore, we get

$$(u^2)^{(1/2)}(1) = -\frac{4}{3\sqrt{\pi}} \ne 0. \tag{31}$$

So we give

$$(u^2)^{(1/2)}(1) \ne 2u(1)u^{(1/2)}(1). \tag{32}$$

Therefore, if we take $v(t) = u(t)$, we have equivalently from (32)

$$(uv)^{(1/2)}(1) \ne u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1). \tag{33}$$

This shows again that Jumarie's formula (2) is not true for the non-differentiable continuous functions.