Chaos, Solitons and Fractals 108 (2018) 94–103

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Complexity and heterogeneity in a dynamic network

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ARTICLE INFO

Article history:
Received 18 September 2017
Revised 20 December 2017
Accepted 16 January 2018

Keywords:
Complexity
Phase transition
Network formation
Network value
Stochastic process
Critical phenomena
Long-range correlations

ABSTRACT

We present an approximate analytical solution for the connectivity of a network model with a “non-simultaneous” linking scheme. This model exhibits node-space correlations in the link distribution, anomalous fluctuations in the time series of the connectivity variable, and a finite-size effect: the maximum number of links occurs away from the critical value of the system parameter. We derive an exact Master Equation for this model in the form of an infinitesimal time-evolution operator. Fluctuations are much more important than the mean-field approximation predicts, which we attribute to the heterogeneity in the model. Finally, we give a sketch of possible real world applications where the value of a network is related to the number of links.

1. Introduction

Complexity is a polymorphous concept, with definitions that vary from one discipline to another. Herein, we will refer to a system as complex if it exhibits spontaneous emergent phenomena over a small range of values of the free parameter(s) of the system. One of our working hypotheses is that complexity is closely tied to heterogeneity. The results below indicate that node heterogeneity is instrumental in determining the degree of interconnectedness in a model for network dynamics.

Our model demonstrates that finite-size effects can be extremely important, especially in systems that display phase transitions. Furthermore, the interplay between risk and profit indicated by the model leads one to the conclusion that there is an optimal size for types of networks that obey the same general principles, for example economic, biological, and sociological groups large and small [1].

Since we work with a network with a dynamic topology, it is the fluctuations and heterogeneity of the network that are most relevant to the behaviors observed. Indeed, at the parameter values at which the network is complex, these fluctuations in the degree of interconnectedness become extraordinarily large, comparable in size with their allowed range.

This emphasis on the dynamical aspects of network properties represents a departure from the standard approaches to studying networks. Most social and financial networks have been studied with an emphasis on their characteristic topological features, especially the *patterns of connection* (often referred to as complexity) between their elements [2,3]. For example, financial economists have largely discussed the benefits of interactions among financial intermediaries. Some degree of interconnectedness is crucial to the proper functioning of financial systems, as no single institution can access the full range of available capital and investment opportunities in the economy. Connections among financial institutions are also assumed to facilitate risk sharing, decrease the uncertainty faced by individual agents, and so allow agents to offer better services to the economy.

On the other hand, complexity is also regarded as a source of system breakdown [4]. In particular, increasing interconnectedness in the market in terms of contracts among financial institutions comes at the price of increasing inaccuracy in the estimation of systemic risk [5]. So, in financial markets, the challenge for market participants, policymakers, and regulators is to find a balance between the benefits of interconnectedness and its potentially harmful destabilizing effects [6,7].

In the present paper, we examine a model from a new network class based on agent preferences, namely preferred degree networks [8], where the number of links continuously fluctuates and the system has a non-trivial steady state distribution. In this class of networks, the system undergoes a phase transition from

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https://doi.org/10.1016/j.chaos.2018.01.024
0960-0779/

a sparsely connected network to a densely connected one. The intermediate critical condition is the condition of maximal population heterogeneity among the nodes and the system shows an abrupt onset of anomalously large fluctuations in the network connectivity, which emphasizes the heterogeneity of the units' preferences. Our analytical results reveal that these large fluctuations are poorly accounted for by the mean-field approximation of the model.

This class of networks differs from the standard literature of pairwise network formation [9] in that a link is not a bilateral compromise between pairs of nodes. An asymmmetry that is in contrast with the statistical symmetry present in other common methods of generation of random graphs, such as via a linking probability matrix.

This class of models is based on the initiative of single units, which according to their preferences can propose to generate or destroy a connection with another node (which can accept or reject the proposition). This change of perspective introduces a dramatic difference in the dynamics of networks displaying the emergent phenomena typical of complex systems.

The model which we analyse has two types of units [10]: generators and destroyers of links. We propose an analytical solution for the equilibrium mean value of the connectivity, which exhibits a phase transition. We provide a closed-form equation for the phase transition thereby characterizing the correspondent critical point.

These results are obtained using a relativity new approach to stochastic processes which makes use of mathematical notations reminiscent of the quantum mechanical formalism [11], and a mean-field approximation to the Fokker–Planck equation for the probability density of links. This approach has been turned out to be crucial, since no purely analytical prediction has been obtained so far for this model, see [8–10,12,13]), where authors have used a more standard approach to stochastic processes. In order to provide a microscopic description of the network, we write an approximate Langevin equation, which allows one to characterize the network in terms of emergent properties at criticality through the study of spatial and temporal correlations. Additionally, we highlight some limitations of the mean-field approximation in capturing the heterogeneity of the nodes in their dynamics of creating and destroying links. These corrections are non-negligible when the system is at its critical point.

As further hallmarks of complexity, we stress the importance of finite-size effects, observing how finite networks produce more links away from the critical condition. Indeed, real-world systems' statistical properties are affected by finite-size and other truncation effects which can play an important role in defining the complexity of networks in terms of the effects of systems in constrained situations such as a limited number of units in the system (i.e., small groups), which is related to coordination issues [14]. This gives rise to a paradigm of emergent properties of groups including the fact that larger team sizes lead to an increasing need for coordination that can limit the efficiency of group members, drawing attention to the optimal connectivity condition as a function of the global size of the network.

The model we describe is an expository model, having the purpose of highlighting and explaining the most crucial mechanisms underlying the phenomena of complex evolving systems as discussed in many disciplines, in particular economics and finance [15–18].

So, without any predictive intention, we set forth an abstract example of a system which gains value according to its interconnectedness, and bears a cost depending on the number of active nodes (i.e., generators of links). The resulting profitability shows a signature of complexity in terms of finite-size network effects: small groups reach a maximal profitability far from the critical point of maximal heterogeneous population, but they tend to suffer less uncertainty of the expected connectivity. As the size of the network increases, we imagine that the system tends to organize itself near a critical point where the network has its maximal profitability; however, this point is also associated with a very high uncertainty (connection volatility). In this state, the system can be more vulnerable to possible systemic failure since it spends some part of its time in an unprofitable state. In terms of social and economic policy, minimizing the importance of heterogeneity also in mathematical terms (by using the mean-field approximation) leads one to drastically underestimate the size of fluctuations at the critical point, which could lead to an underestimation of the risk in the system.

2. The generators–destroyers model and its analytical description

Among the possible models in the class of unit-driven networks, we select the most simple heterogeneous case where we have a bipartite graph in which two types of nodes exist: one group of nodes aims to create new links every time they are selected, the other group aims to cut a link with previously connected partners.

The Generators–Destroyers model is a model for the intergroup link dynamics between a group of link generators and a group of link destroyers directly derived from the introvert-extrovert model (XIE) as introduced and studied in [8,10]. In the following calculations, it is assumed that time is continuous (in the sense that there is no fixed minimum time between changes in the number of links), one can think of this as an event-based approach. Links are bi-directional, and there can be at most one link between any two units. Neither self-links nor intragroup links are considered, as these subgraphs quickly go to a static equilibrium state, thus the graphs produced come from a subset of the set of simple graphs. The generators create links as long as there is at least one destroyer available to which it has no link. Destroyers destroy links until they are not linked to any generators.

It is convenient to represent a graph in terms of a matrix, called the adjacency matrix. This matrix is formed by enumerating the vertices of the graph, the i, jth entry of the matrix is the number of links from vertex i to vertex j. In the model under consideration, the links are bi-directional, and there is at most one for every pair of vertices, so the adjacency matrix is symmetric (a bi-directional link consists of one unidirectional link in each direction) and consists of ones and zeros (either a link is occupied or it is not). Since the graph is dynamical in the generators–destroyers model, so is its associated adjacency matrix.

As mentioned in the introduction, since the standard tools for studying stochastic processes have not sufficed to find an analytical solution for the phase transition of the present system, as a possible path towards an analytical solution to this model, the authors found useful to give to the dynamics of the system a physical interpretation in terms of an operator formalism like that used for the harmonic oscillator in quantum mechanics. In the following theoretical treatment, we model the dynamics as due to the action of an infinitesimal stochastic time-evolution operator H on the adjacency matrix. One can imagine such an operator as a sum of simpler operators, one for each element of the adjacency matrix.

In the present mathematical description we focus on an analytical treatment in terms of the average number of links and link distribution, instead of the degree-distribution and average degree as in [9]. In order to derive an equation of motion for the link distribution, we write a Fokker–Planck equation for this model starting from the formalism of creation and annihilation operators whose basic notions are elucidated in Appendix A. The only free parameter in this representation is the total rate of events, i.e., an overall