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Extreme multi-stability: When imperfection changes quality

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ABSTRACT

In this paper, we discuss how chaotic systems show the importance of imperfection. This happens through the butterfly effect. Then we discuss that chaotic systems with extreme multi-stability can much better demonstrate such importance. The reason is that in such systems not only the quantity of time-series is affected by butterfly effect, but also the quality of time-series is changed by small imperfections in parameters or initial conditions. We prove the importance of that difference better by comparing the efficiency of a newly proposed parameter estimation method on both an ordinary chaotic system and a chaotic system with extreme multi-stability.

1. Introduction

The study of chaotic systems has received increasing attention and has provided a promising method for studying many real-world systems [1–4]. Chaos occurs in dynamical systems that are sensitively dependent on initial conditions. Small differences in initial conditions yield widely diverging outcomes. This sensitivity has been called butterfly effect [5]. It means that when you want to predict a chaotic time-series, even if you have a super-accurate model of the system generating that time-series, you cannot do an accurate prediction for a long time. Any tiny imperfections in the model's parameters or initial conditions cause a huge error between estimated time-series and real ones. However such difference appears only in the quantity of time-series, and not in their quality. It means that although the time-series are different in time, they have the same attractors in state space. Consider the well known Lorenz system [6] which is described in Eq. (1).

$$\begin{aligned} &\dot{x} = \sigma(y - x) \\ &\dot{y} = x(\rho - z) - y \\ &\dot{z} = xy - \beta z \\ &\sigma = 10,\ \beta = 8/3,\ \rho = 28 \end{aligned} \tag{1}$$

Fig. 1 shows the effect of 5% change in the initial conditions in the Lorenz system. While the butterfly effect is observable from part (a) of this figure, part (b) indicates that both initial conditions converge to the same attractors.

In recent years some new dynamical systems have been introduced which are called systems with extreme multi-stability [7–9]. In those systems, any imperfect estimation of parameters and initial conditions not only cause time-series with different quantities, but also with different qualities. It means that a tiny change in the initial conditions may result in two different attractors in the state space.

For example consider the system in Eq. (2).

$$\begin{aligned} &\dot{x} = -ay \\ &\dot{y} = c\left(b_1 - b_2 x^2\right)y - c(y - z) \\ &\dot{z} = -ky + k(1 + r_1)z - \tfrac{(2k+1)r_2}{k+1} w \\ &\dot{w} = -(k+1)y + (k+1)(1 + r_1)z - 2r_2 w \\ &a = 3,\ b_1 = 15/14,\ b_2 = 3/28, c = 20,\ r_1 = 15,\ r_2 = 0.15, \\ &k = 0.05 \end{aligned} \tag{2}$$

System (2) is a dynamical system with extreme multi-stability as reported in [7]. Fig. 2 shows the effect of 5% change in the initial conditions in this system. While the butterfly effect is still observable from part (a) of this figure, part (b) indicates that those initial conditions result in completely different attractors.

2. Parameter estimation test

Most control, synchronization, and image encryption methods related to chaotic systems need the parameters of the considered system to be determined. It is often difficult to measure the system's parameters directly. Fortunately, usually some time-series of a chaotic system can be recorded experimentally. By the help of obtained time-series, the problem of parameter estimation can be

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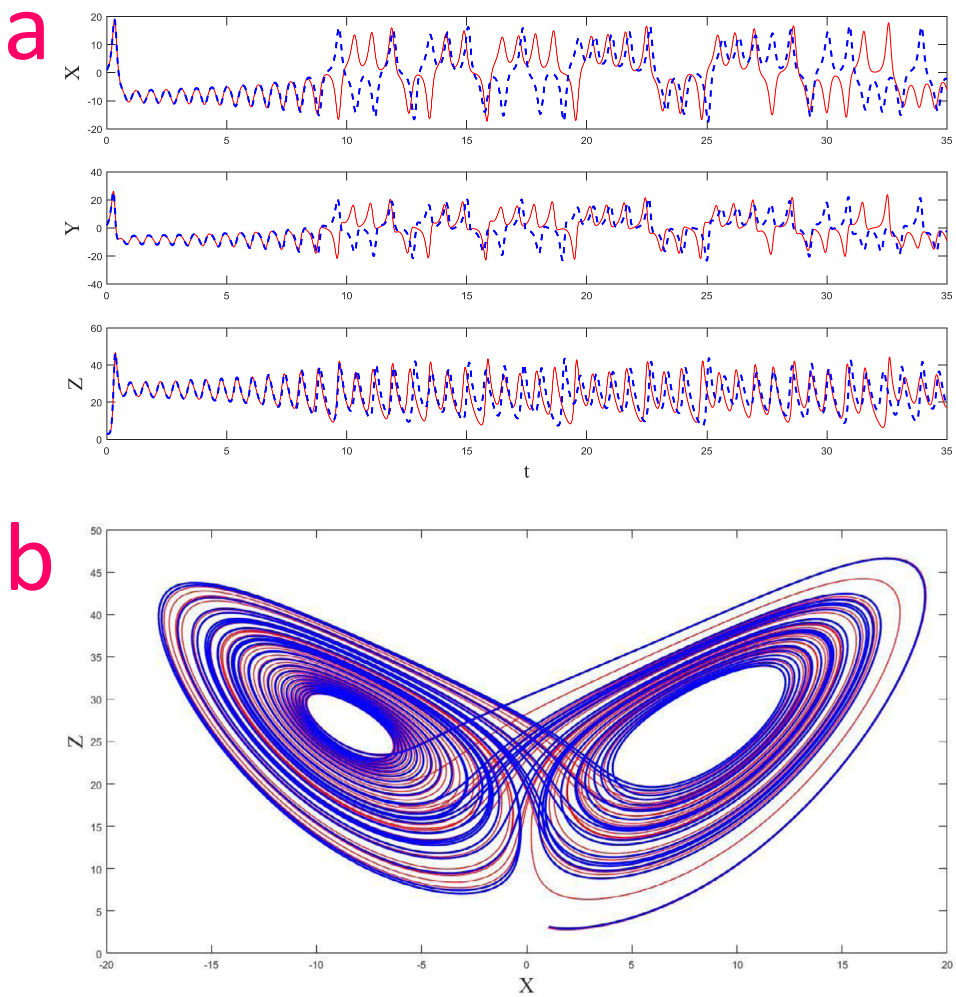

Fig. 1. Two Chaotic evolution of the Lorenz system with parameters given in Eq. (1) showing the effect of 5% difference in the initial conditions. The initial conditions are $(x_0, y_0, z_0) = (1, 2, 3)$ and $(x_0, y_0, z_0) = (1.05, 2.1, 3.15)$. a) butterfly effect in time-series and b) similar attractors in state space.

formulated as a cost function which should be minimized [10–13]. Such methods define their objective functions based on similarity between a time-series obtained from the real system and model in the time domain. It has been shown that those methods bear some major limitations due to butterfly effect [14–16].

Although chaotic systems have random-like behavior in the time domain, they are ordered in state space and have a specific topology. Based on this fact, a new cost function (geometry-based cost function) has been proposed in [17]. In that method, in summary, a return map is constructed based on one real observed time-series. Then a similar procedure is done for the model data. Finally the following algorithm is performed:

I. For each point of the real data, its nearest neighbor in the model set of points is found and their Euclidean distance is calculated.
II. For each point of the model data, its nearest neighbor in the real set of points is found and their Euclidean distance is calculated.
III. The cost function is taken as the average of those distances over the whole data set.

For the complete details see [17].

Consider System (1) as a real system in which we don't know the value of the parameter ρ. We assume that we only have access to the time-series of x(t) from System (1).

The cost function for this case is shown in Fig. 3 along with the bifurcation diagrams of the system. As can be seen, this cost function has the ideal properties. It shows the effect of changing the parameter of the model, including the bifurcations and the monotonic trend along with a global minimum at the right value ($\rho = 28$).

Now consider System (2) as a real system in which we don't know the value of parameter k. We assume that we only have access to the time-series of x(t) from it. The cost function for this case is shown in Fig. 4 along with the bifurcation diagrams of the system. As can be seen, the cost function is not proper at all. Even it doesn't have global minimum at the right place ($k = 0.05$). This is because of this fact that in systems with extreme multi-stability, the butterfly effect occurs not only in time domain, but also in state space. It means that it is near impossible to estimate the parameters of the chaotic system with extreme multi-stability, which make them very proper candidates for e.g. being used in secure communications methods [18–20].

3. Conclusion

We demonstrated that how chaotic systems show the importance of imperfection through the butterfly effect. Then we showed that chaotic systems with extreme multi-stability can much better demonstrate such importance. In such systems, tiny difference in the initial conditions not only the change the quantity of time-