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Nonlinear electroelastostatics: Incremental equations and stability

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ABSTRACT

In this paper we first provide an overview of the recently formulated nonlinear constitutive framework for the quasi-static response of electroelastic solids and its isotropic specialization. The general theory exhibits a strong nonlinear coupling between electric and mechanical effects. The main part of the paper focuses on the governing equations describing the linearized response of electroelastic solids superimposed on a state of finite deformation in the presence of an electric field for independent incremental changes in the electric displacement and the deformation within the material. The associated incremental changes in the stress and the electric field within the material and the surrounding space and the incremental boundary conditions are derived for mechanically unconstrained and constrained electroelastic solids and in the isotropic specialization. By way of illustration of the incremental theory, we specialize the constitutive law to an electroelastic neo-Hookean material, and consider the stability of a half-space subjected to pure homogeneous deformation in the presence of an applied electric field normal to its surface. We show that stability is crucially dependent on the magnitudes of the electromechanical coupling parameters in the constitutive equation.

1. Introduction

The theory of nonlinear electroelasticity accounts for the coupling of electrical and mechanical material properties of electro-active solids subjected to finite strain. The theory, originally developed by Toupin [25] in 1956, has seen a revival of interest recently because the electromechanical coupling of these materials opens the door for the development of many new devices, impacting a range of applications that could not be addressed with previously available materials [1,21].

Electro-active elastomers, in particular, are materials that rapidly and reversibly change their mechanical properties in response to the application of an electric field. Typically, the electromechanical coupling is achieved and optimized by mixing, during the vulcanization process, nano- or micron-sized polarizable particles within a soft and highly elastic matrix material. The change in mechanical and electrical properties is attributed to the interactions between neighbouring particles within the matrix. Preferred matrix materials include gels, resin, natural rubber, silicone rubber and nitrile rubber, and the particles of choice are spherical or irregularly shaped carbonyl iron particles [3,9]. Under the effect of an externally applied electric field these mechanically soft materials are capable of large elastic deformations that are much larger than those arising in conventional electrostriction.

This nonlinear electromechanical coupling in electro-active elastomers has generated much interest since the original publication by Toupin [25]. Relevant background information is provided in the articles by Truesdell and Toupin [26] and Tiersten [24] and in the book by Landau and Lifschitz [16]. Materials are now available that can operate in a highly nonlinear

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electromechanical regime and they offer exciting possibilities from the perspective of constitutive modelling and electromechanical theory. This interest is evidenced by a range of books dealing with the nonlinear interaction between mechanical and electromagnetic fields (Maugin [17], Eringen and Maugin [12], Kovetz [15] and Hutter et al. [14], for example). Recent journal articles include those by Ericksen [10,11], McMeeking and Landis [18], McMeeking et al. [19], Suo et al. [23], Steigmann [22], Fosdick and Tang [13] and by the authors of this paper [4,7,8].

In Section 2, following Dorfmann and Ogden [7,8], we provide a brief overview of the basic electrical and mechanical balance laws for time-independent electric fields. We then give the general constitutive law for an isotropic electroelastic material based on a total energy function that enables expressions for the stress and electric field variables to be cast in particularly simple forms. Two alternative formulations are highlighted in [7]. In one formulation the deformation gradient and the applied electric field vector are taken as the independent variables, while in the other the electric field is replaced by the electric displacement vector. In the present work we focus on the latter formulation with the energy density treated as a function of the deformation gradient and the electric displacement vector, leading to explicit expressions for the total stress tensor and the electric field vector in both Eulerian and Lagrangian forms. Appropriate boundary conditions are specified for the electric field variables and for the total stress tensor.

In Section 3 we consider incremental changes in the deformation within the material and in the electric displacement vector both within the material and its environment. The governing equations for the associated incremental changes in the total stress and the electric field within the material and the surrounding space and the incremental boundary conditions are derived for unconstrained and incompressible electroelastic materials. The incremental equations require second-, third- and fourth-order electroelastic moduli tensors, which are derived for an isotropic material and listed in compact form. A parallel analysis for magnetoelastic materials is contained in the recent paper by Otténio et al. [20].

The basic electroelastic theory of Section 2 and the incremental equations of Section 3 are then used to evaluate the surface stability of an electroelastic half-space. In Section 4 the components of the total stress and the electric field in a half-space subjected to pure homogeneous deformations in the presence of an applied electric field normal to its free surface are given, and appropriate boundary conditions are used to obtain the (Maxwell) stress and electric field components outside the material. In Section 5, the general incremental equations are applied to the analysis of surface stability. In total there are 7 homogeneous linear equations for 7 unknowns and a bifurcation criterion is obtained by setting the 7×7 determinant of coefficients equal to zero. Numerical solutions are then obtained for a simple prototype model, namely a neo-Hookean electroelastic material. In addition to the standard shear modulus, the energy function includes two electromechanical coupling parameters. The stability of the electroelastic half-space depends critically on their magnitudes and on the magnitude of the applied electric field. It is shown that for selected values of the parameters, an increasing electric field has a stabilizing effect, but for different values it has a destabilizing influence. The half-space may become unstable when subjected to either tension or compression parallel to its surface.

2. The equations of nonlinear electroelasticity

In this section we summarize the form of the equations for nonlinear electroelastic deformations given by Dorfmann and Ogden [7,8] as a basis for the derivation of the incremental equations in Section 3.

We consider an electroelastic body in a stress-free undeformed *reference configuration* $\mathcal{B}_0$, with boundary $\partial\mathcal{B}_0$. In $\mathcal{B}_0$ material points are labelled by their position vectors $\mathbf{X}$. The material is deformed quasi-statically from $\mathcal{B}_0$ to the *deformed configuration* $\mathcal{B}$ (with boundary $\partial\mathcal{B}$) as a result of applied mechanical loads and an applied electric field. The deformation is described by the vector function $\boldsymbol{\chi}$, and $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ denotes the position of $\mathbf{X}$ in $\mathcal{B}$. It is assumed that $\boldsymbol{\chi}$ is sufficiently well behaved for our purposes. The deformation gradient tensor $\mathbf{F}$ is defined by $\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}$, Grad being the gradient operator in $\mathcal{B}_0$, and has Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$. Roman indices are associated with $\mathcal{B}$ and Greek indices with $\mathcal{B}_0$. We adopt the standard notation $J = \det\mathbf{F}$, with the convention $J > 0$. The left and right Cauchy-Green tensors associated with $\mathbf{F}$ are denoted here by $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, respectively, where $^{\mathrm{T}}$ denotes the transpose of a second-order tensor.

2.1. Mechanical equilibrium

For a compressible material J is a local measure of the change in mass density, from ρ_0 in $\mathcal{B}_0$ to ρ in $\mathcal{B}$, via the equation

$$J\rho = \rho_0. \tag{1}$$

For an incompressible material, $\rho = \rho_0$ and the incompressibility constraint $J = 1$ is enforced.

In the absence of mechanical body forces, equilibrium is maintained through the equation

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \tag{2}$$

coupled with appropriate boundary conditions (to be discussed in Section 2.3), where $\boldsymbol{\tau}$ is the *total Cauchy stress tensor*, which is *symmetric*, and div is the divergence operator in $\mathcal{B}$. The *total nominal stress tensor* $\mathbf{T}$ is then defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \tag{3}$$