Chaos, Solitons and Fractals 106 (2018) 1–4

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

On the global dynamics of a finance model

Jaume Llibre[a], Clàudia Valls[b,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

[b] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

ARTICLE INFO

Article history:
Received 14 October 2017
Accepted 25 October 2017

JEL classification:
34C05
34C07
34C08

Keywords:
Darboux invariant
Finance model
Poincaré compactification
Global dynamics

ABSTRACT

Recently several works have studied the following model of finance

$\dot{x} = z + (y - a)x, \quad \dot{y} = 1 - by - x^2, \quad \dot{z} = -x - cz,$

where a, b and c are positive real parameters. We study the global dynamics of this polynomial differential system, and in particular for a one–dimensional parametric subfamily we show that there is an equilibrium point which is a global attractor.

1. Introduction and statement of the results

We consider the following polynomial differential system in $\mathbb{R}^3$ (see [1–3,5–7])

$$\begin{aligned} \dot{x} &= z + (y - a)x, \\ \dot{y} &= 1 - by - x^2, \\ \dot{z} &= -x - cz, \end{aligned} \tag{1}$$

where the parameters a, b, $c > 0$. This model describes the time variation of three state variables: the interest rate x, the investment demand y and the price index z. Here a is the saving amount, b is the cost per investment and c is the elasticity of demand of commercial market. The factors that influence changes in x mainly come from an excess of investment over savings and the structural adjustment from good prices. Changing rates in y are in proportion to the rate of investment and in proportion to an inversion with the cost of investment and interest rates. Changes in the variable z are controlled by a contradiction between supply and demand and are influenced by inflation rates.

In this paper we shall provide the complete description of the global dynamics of this polynomial differential system not only on $\mathbb{R}^3$ but also in its compactification for some values of the parameters. In this way we will also control the orbits which come or go to infinity. So, this complete information about the dynamics of the polynomial differential system (1) will help to a better understanding of it. More precisely, we want to describe the α-limit sets and the ω-limit sets of all orbits of system (1) for some values of the parameters. Let $\varphi(t) = \varphi(t, p)$ be the solution of system (1) passing through the point $p \in \mathbb{R}^3$ when $t = 0$, defined on its maximal interval $I_p = (\overline{\alpha}(p), \overline{\omega}(p))$. If $\omega(p) = \infty$, then the *ω-limit* set of φ is

$$\omega(\phi) = \{q \in \mathbb{R}^3 : \exists \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}$$

Similarly, if $\overline{\alpha}(p) = -\infty$ then the *α-limit* set of φ is

$$\alpha(\phi) = \{q \in \mathbb{R}^3 : \exists \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}$$

For more characterizations of the α-limit set and of ω-limit set of an orbit, see for instance Section 1.4 of [4].

Note that system (1) is defined in the open manifold $\mathbb{R}^3$. For studying its orbits in a neighborhood of the infinity (which has to be done if one wants to study the α and ω-limit sets of the system) we shall identify $\mathbb{R}^3$ with the interior of the unit ball

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$$

centered at the origin, and we shall extend analytically this flow to its boundary $\mathbb{S}^2$ (the infinity). This compactification is due to Poincaré, and this ball is called *the Poincaré ball*. The polynomial differential system (1) extended to this closed ball is called *the Poincaré compactification of the polynomial differential system (1)*. For a precise definition of all these notions see the appendix. We shall note that the extended flow to the Poincaré ball leaves invariant the boundary of the ball (the infinity) in the sense that if

* Corresponding author.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

https://doi.org/10.1016/j.chaos.2017.10.026
0960-0779/

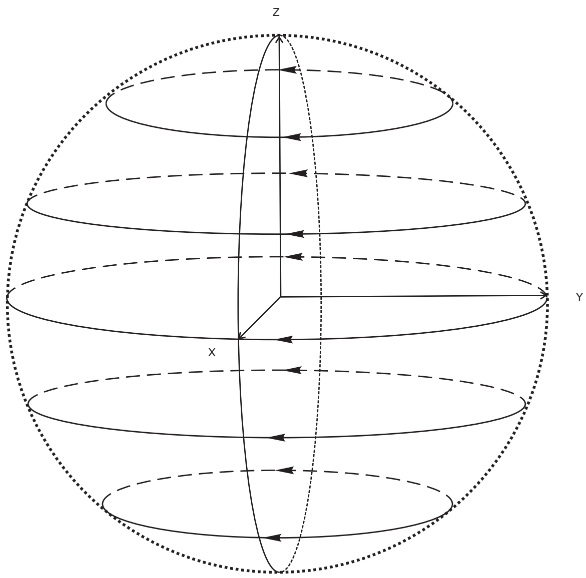

Fig. 1. Phase portraits at infinity of system (1). The boundary at infinity of the plane $x = 0$.

an orbit has a point in this boundary then the whole orbit is contained in it.

As explained above the infinity is invariant by the Poincaré compactification of system (1). The flow on this boundary (a two dimensional sphere $\mathbb{S}^2$) is described in the next theorem. For a definition of topologically equivalent phase portraits, see for instance section 1.3 of [4].

Theorem 1. *The phase portrait of the Poincaré compactification of system (1) at the infinity $\mathbb{S}^2$ is topologically equivalent to the one described in Fig. 1.*

Let $\mathbb{R}[x, y, z]$ be the ring of real polynomials in the variables x, y, z. We say that $F = F(x, y, z)$ is a *Darboux polynomial* of system (1) if it satisfies

$$\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial z}\dot{z} = kF$$

where $k = k(x, y, z)$ is a real polynomial of degree at most one called the *cofactor* of F and $(\dot{x}, \dot{y}, \dot{z})$ are given in (1). If the cofactor of F is zero, then $F(x, y, z)$ is a *polynomial first integral* of system (1). If $F(x, y, z)$ is a Darboux polynomial then the surface $F(x, y, z) = 0$ is an invariant algebraic surface.

We say that a C^1 function $I(x, y, z, t)$ is an *invariant* of the differential system (1) if $I(x, y, z, t)$ is constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined. When an invariant function is independent of the time t, then it is a *first integral*.

When a system has a Darboux polynomial F with a constant factor $k = k_0 \in \mathbb{R}$, then the function $I(x, y, z, t) = F(x, y, z)e^{-k_0 t}$ is called a *Darboux invariant* of that system, see Chapter 8 of [4].

Theorem 2. *System (1) with*

$$a = -\frac{4 + k_0^2}{2k_0}, \quad b = -\frac{k_0}{2}, \quad c = -\frac{k_0}{2}, \tag{2}$$

where $k_0 < 0$ has the Darboux polynomial

$$F_{k_0}(x, y, z) = k_0^2 x^2 + k_0^2 z^2 + (2 + k_0 y)^2$$

and the Darboux invariant

$$I_{k_0}(x, y, z, t) = \left(k_0^2 x^2 + k_0^2 z^2 + (2 + k_0 y)^2\right)e^{-k_0 t}.$$

For the values of the parameters given in (2) we have that $F_{k_0} = 0$ is an invariant algebraic surface.

For the values (a, b, c) given in (2), system (1) can be written as

$$\begin{aligned} \dot{x} &= \frac{4 + k_0^2}{2k_0}x + z + xy, \\ \dot{y} &= 1 + \frac{k_0 y}{2} - x^2, \\ \dot{z} &= -x + \frac{k_0 z}{2}, \end{aligned} \tag{3}$$

where $k_0 < 0$. In the following theorem we describe the dynamics of system (3). We recall that a point p is *globally asymptotically stable* for system (3) if every solution $(x(t), y(t), z(t))$ of system (3) is defined for $t \to \infty$ and tends to p when $t \to \infty$.

Theorem 3. *For the values of the parameters (a, b, c) given in (2) the phase portrait of system (3) in $\mathbb{R}^3$ is as follows: the invariant algebraic surface $F_{k_0}(x, y, z) = 0$ is formed by the point $q = (0, -2/k_0, 0)$, which is the unique equilibrium point of system (3), and it is globally asymptotically stable.*

In the next theorem we describe the α- and the ω-limit sets of all orbits contained in B of system (3). Let P be a diffeomorphism such that $P(\mathbb{R}^3)$ is equal to the interior of the Poincaré ball. Then we denote the finite isolated singular point q of system (3) given in Theorem 3 as $p = P(q)$.

Theorem 4. *Let γ be an orbit of system (3).*

(a) *If γ is contained in the boundary of the Poincaré ball B and is different from a singular point, then its α- and ω-limit sets are different infinite singular points.*

(b) *If γ is contained in the interior of B and is different from the singular point $q = (0, -2/k_0, 0)$, then the following two statement hold.*

(b.1) *The α-limit set of γ is an infinite singular point.*

(b.2) *The ω-limit set of γ is the unique finite singular point q.*

Note that in both cases such a γ is a heteroclinic orbit.

The proofs of all theorems are given in the next section.

We remark that Theorem 1 works for system (1), i.e. for the system with the three parameters a, b and c. While the other theorems only work for the one-dimensional subfamily of systems (1) defined by condition (2).

2. Proof of the results

Proof of Theorem 1. For studying the infinity of the Poincaré ball B we analyze the flow at infinity for the local charts U_1, U_2 and U_3, see the appendix.

In the local chart U_1 system (1) writes

$$\begin{aligned} \dot{z}_1 &= -1 - z_1^2 + (a - b)z_1 z_3 + z_3^2 - z_1 z_2 z_3, \\ \dot{z}_2 &= -z_3 - z_1 z_2 + (a - c)z_2 z_3 - z_2^2 z_3, \\ \dot{z}_3 &= z_3(-z_1 + a z_3 - z_2 z_3). \end{aligned} \tag{4}$$

System (4) restricted at the infinity (that is with $z_3 = 0$) becomes

$$\dot{z}_1 = -1 - z_1^2, \quad \dot{z}_2 = -z_1 z_2.$$

so there are no singular points at infinity in the local chart U_1.

In the local chart U_2 system (1) writes

$$\begin{aligned} \dot{z}_1 &= z_1 - (a - b)z_1 z_3 + z_2 z_3 + z_1^3 - z_1 z_3^2, \\ \dot{z}_2 &= -z_1 z_3 + (b - c)z_2 z_3 + z_1^2 z_2 - z_2 z_3^2, \\ \dot{z}_3 &= z_3(b z_3 + z_1^2 - z_3^2). \end{aligned} \tag{5}$$

System (5) restricted at the infinity (that is with $z_3 = 0$) becomes

$$\dot{z}_1 = z_1(1 + z_1^2), \quad \dot{z}_2 = z_1^2 z_2,$$