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A hidden chaotic attractor in the classical Lorenz system

Buncha Munmuangsaen*, Banlue Srisuchinwong

Sirindhorn International Institute of Technology, Thammasat University, 131 Moo 5, Tiwanont Road, Bangkadi, Mueang, Pathum Thani 12000, Thailand

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A B S T R A C T

Chaotic attractors in the classical Lorenz system have long been known as self-excited attractors. This paper, for the first time, reveals a novel hidden chaotic attractor in the classical Lorenz system. Either a self-excited or a hidden chaotic attractor is now possible in the classical Lorenz system depending on values of both system parameters and initial conditions. A systematically exhaustive computer search is employed to directly search for the hidden chaotic attractor with elegant values of both system parameters and initial conditions. Time series of trajectories, Lyapunov exponents, and bifurcations of the hidden chaotic attractor are reported. Basins of attraction of individual equilibria are depicted to verify that the hidden chaotic attractor is found. Dynamic regions of attractors are illustrated to reveal seamless connections between self-excited and hidden chaotic attractors in the classical Lorenz system with wide ranges of parameters.

1. Introduction

Since 1963, an era of chaos has been opened by a young meteorologist Edward Lorenz who accidentally discovered the celebrated Lorenz system in a set of three coupled first-order ordinary differential equations (ODEs) [1]. His discovery has stimulated others to explore more chaotic systems (e.g. Rössler [2], jerk [3,4], Circulant [5,6], hyperjerk [7,8], hyperchaotic [7,9,10] systems) and chaotic circuits (e.g. Lorenz-based chaotic circuits [11,12], Chua's circuits [13–15], Wien-type chaotic oscillator [16], chaotic jerk circuits [17–20]). Chaos theory has therefore been an important branch of nonlinear dynamics, modern physics, mathematics and engineering. Studies of chaos have increasingly attracted much attention due to its possible applications in various fields of science and technology.

Since 2010, the first hidden chaotic attractor [21] has been discovered in a generalized Chua's system. Attractors are therefore classified as self-excited and hidden attractors [22,23]. The classification involves its basin of attraction, which refers to a set of initial conditions whose trajectories tend to the attractor. An attractor is called a *self-excited attractor* if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise it is called a *hidden attractor* [24]. Multistability in hidden attractors can cause major problems to engineering and industry applications, and may allow unexpected disasters in structures such as drilling rigs failures [25] and bridge collapses [26], or allow catastrophic events such as the crash of the aircraft YF-22 Boeing in 1992 due to the sudden shift to the undesired attractor [27]. It is therefore very important to reveal all possible attractors in a system, such that appropriate controlling methods can be applied to prevent such serious occurrences [28,29].

An analytical-numerical algorithm has been proposed by Leonov et al. [21,23] to localize an appropriate initial condition for a hidden attractor. The algorithm however requires a combination of analytical and numerical procedures, which are applicable to only a system of potentially periodic oscillation. In addition, the algorithm is not systematic due to some random adjustments which depend upon designer's experiences and talents [30]. Examples of hidden attractors based on this algorithm can be found in [23,24,31].

On the other hand, Sprott et al. [30,32–37] have employed a traditionally exhaustive search for chaos in some unusual chaotic systems of special equilibria where the chaotic attractors are always known to be hidden. Examples of such unusual systems include chaotic systems with no equilibrium [32], one stable equilibrium [33], a single unstable node [34], a line equilibrium [35], a plane of equilibria [30], a square equilibrium [36], or surfaces of equilibria [37]. The traditional search for chaos in these unusual systems, however, cannot be used to detect and separate hidden attractors from self-excited attractors in a typical system where its attractors can be self-excited or hidden attractors depending on values of initial points and system parameters, such as the classical Chua's system and the classical Lorenz system.

Recently, Munmuangsaen et al. [38] has developed a new computer methodology of a systematically exhaustive search for hidden and self-excited attractors in a typical system that may have

* Corresponding author.
E-mail addresses: buncha.munmuangsaen@studentmail.siit.tu.ac.th, nopnop99@hotmail.com (B. Munmuangsaen).

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an unstable saddle equilibrium. Such a search for hidden attractors has been applied to the classical Chua's system, and three new hidden attractors have been revealed. Although a hidden attractor in the generalized Lorenz system has recently been reported [24], a question has been raised whether a hidden attractor is possible in the classical Lorenz system. It is a challenge whether the new computer methodology of the exhaustive search for hidden attractors developed in [38] will be able to detect and separate a hidden attractor from a self-excited attractor in the classical Lorenz system. What follows is the answer.

2. A novel hidden chaotic attractor in the classical Lorenz system

The classical Lorenz system is described by a set of three coupled first-order ODEs in dimensionless coordinates as [1,11,12]

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

The usual values of system parameters originally introduced by Lorenz are $a = 10$, $r = 28$ and $b = 8/3$, which have produced a self-excited chaotic attractor in a butterfly shape. The system (1) has three equilibrium points as

$$\begin{aligned} S_1 &= (0, 0, 0) \\ S_2 &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1)) \\ S_3 &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1)). \end{aligned} \tag{2}$$

It follows from (2) that both system parameters b and r determine the stability of the equilibrium points, whereas the system parameter a does not. It is therefore reasonable that both parameters b and r are adjustable, whereas the parameter a will be left as a fixed constant.

The systematically exhaustive search for hidden attractors developed in [38] is applied to system (1). The major difference from [38] is that the initial conditions used in this paper are chosen from a Gaussian distribution with mean = 0 and variance = 10. The reason is to expand the search fields due to the fact that the attractor size of the Lorenz system is typically 10 times larger than the attractor size of the classical Chua's system. As a result, newly found system parameters are, for example, $a = 4$, $r = 29$ and $b = 2$, and a newly found initial condition is, for example, $L_1 = (x_0, y_0, z_0) = (5, 5, 5)$. Such newly found values of the parameters and initial condition are elegant in the sense that they are all integers.

For $a = 4$, $r = 29$ and $b = 2$, it follows from (2) that three equilibrium points are at $S_1 = (x_1, y_1, z_1) = (0, 0, 0)$, $S_2 = (x_2, y_2, z_2) = (-7.4833, -7.4833, 28)$, and $S_3 = (x_3, y_3, z_3) = (7.4833, 7.4833, 28)$. At the equilibrium point S_1, the corresponding eigenvalues are $(-13.3743, 8.3743, -2)$. Since it has one positive real eigenvalue and two negative real eigenvalues, the equilibrium point S_1 is therefore an unstable saddle point with index 1 where the index refers to the number of eigenvalues whose real parts are positive. At the equilibrium points S_2 and S_3, the corresponding eigenvalues are $(-6.8764, -0.0618 \pm 8.0713i)$. Since they have one real eigenvalue and a pair of complex conjugate eigenvalues, with all negative real parts, the equilibrium points S_2 and S_3 are therefore stable focus-node points.

2.1. A hidden chaotic attractor and point attractors

Fig. 1 illustrates a newly found hidden chaotic attractor in green on an (x, y) plane using the initial condition $L_1 = (x_0, y_0, z_0) = (5, 5, 5)$. In addition, Fig. 1 also includes two point attractors in blue and red, which converge on the stable equilibrium points S_2 and S_3, respectively. An initial condition of the blue point attractor for S_2 is at $L_2 = (x_0, y_0, z_0) = (0.1, 0, 0)$, whereas an initial condition of the red point attractor for S_3 is at $L_3 = (x_0, y_0, z_0) = (-0.1, 0, 0)$. Fig. 2 alternatively displays all attractors in Fig. 1 on an (x, z) plane. The spectrum of Lyapunov exponents (LEs) of the hidden chaotic attractor in green is $(\lambda_1, \lambda_2, \lambda_3) = (0.6707, 0, -7.6707)$, calculated with 10^7 iterations (to ensure that chaos is neither numerical artifacts nor chaotic transients). The positive Lyapunov exponent confirms chaoticity of the found hidden chaotic attractor. The Kaplan–Yorke dimension is $D_{KY} = 2.0874$. The correlation dimension based on the method described in [39] is $D_C = 2.0895 \pm 0.01$.

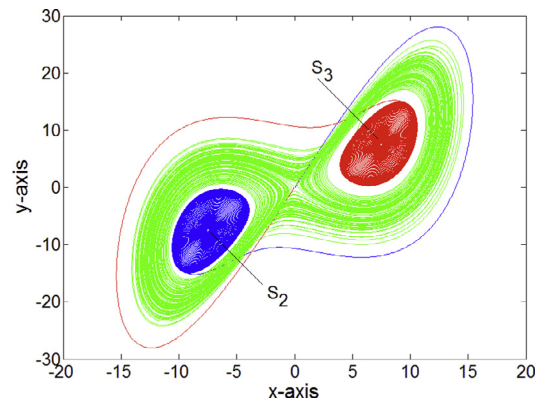

Fig. 1. A new hidden chaotic attractor (green) and two point attractors (blue and red) on an (x, y) plane of the classical Lorenz system. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

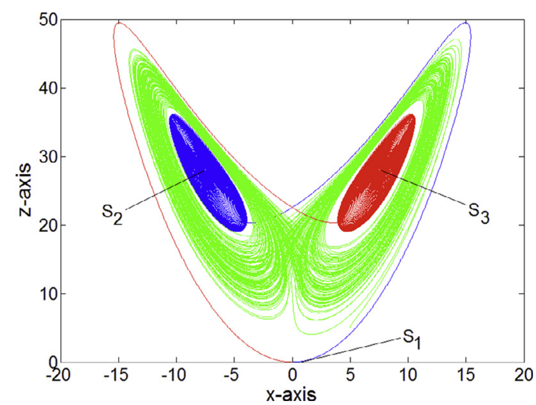

Fig. 2. A new hidden chaotic attractor (green) and two point attractors (blue and red) on an (x, z) plane of the classical Lorenz system. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 3 illustrates a green time series of the hidden chaotic attractor, and two red and blue time series of the two point attractors. It can be seen from Fig. 3 that the green time series of the hidden chaotic attractor swings between positive and negative values of x, and appears in a similar manner to the time series of the best known butterfly-shaped self-excited Lorenz attractor. By contrast, the blue and red time series of the two point attractors are displayed in the negative and positive values of x, which are gradually attracted into the stable focus-node points S_2 and S_3, respectively.

2.2. A smooth transition from a self-excited to a hidden chaotic attractor

For $1.8 < b \leq 1.985$, the three colored attractors in Figs. 1 to 3 become three colored self-excited chaotic attractors. Fig. 4 is separated into two areas, an upper-half area (UHA) and a lower-half area (LHA), each of which illustrates three colored plots in