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Approximation methods for the stability analysis of complete synchronization on duplex networks

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ABSTRACT

Recently, the synchronization on multi-layer networks has drawn a lot of attention. In this work, we study the stability of complete synchronization on duplex networks. We first numerically investigate the effects of coupling functions on complete synchronization on duplex networks. Then, we propose two approximation methods to deal with the stability of complete synchronization on duplex networks. In the first method, we introduce a modified master stability function and, in the second method, we only take into consideration the contributions of a few most unstable transverse modes to the stability of complete synchronization. We find that both methods work well for predicting the stability of complete synchronization for small networks. For large networks, the second method still works pretty well.

1. Introduction

Network science has provided a fertile ground for understanding complex systems. The traditional network approach treats complex systems as monolayer networks by charting an elementary unit into a network node and representing each unit-unit interaction on an equivalent footing as a network link. [1–7]. Recently, it has become clear that structures of many complex systems in social, biological, technological systems should not be treated as monolayer networks but multi-layer ones [8–17]. There are different types of multi-layer networks, such as networks of networks, interacting networks, multiplex networks. Consider a group of members, in which every member may interact with the others through different channels such as Twitter, blog, Facebook, and Wechat. The social network formed by these members is a typical example of multiplex networks, in which different interaction channels are represented as different layers. Following the terminology in the Refs. [17–20], a multiplex network consists of several layers, each of them characterized by a distinct interaction channel, and all layers share the same nodes.

Synchronization, one of the most interesting collective behaviors, has been investigated since the dawn of natural science [5,21–24]. There are different types of synchronization on monolayer networks or multi-layer networks [17], such as complete synchronization (CS) [5], phase synchronization [25], lag synchronization [26], general synchronization [27], cluster synchronization [28], partial synchronization [29], and remote synchronization [30]. Among all types of synchronization, CS where the states of all oscillators are identical is the simplest one [31]. CS on monolayer networks may be well studied by using the master stability function (MSF) method [32,33]. The MSF method shows that the stability of CS is affected by coupling functions (CFs) and network structures.

Recently, much attention has been paid to synchronization on multi-layer networks [15,34–37], especially on multiplex networks [16,38]. Aguirre *et al.* have shown that connecting the high-degree (or low-degree) nodes in different layers turns out to be the most (or the least) effective strategy to achieve synchronization in interacting networks [15]. Using the MSF method, Sorrentino *et al.* have studied CS on duplex networks when the two layers are subject to constrains such as commuting Laplacians, unweighted and fully connected layers, and nondiffusive coupling [16]. Genio *et al.* have provided a full mathematical framework to evaluate the stability of CS on multiplex networks by generalizing the MSF method [38]. However, $N-1$ transverse modes are coupled in their framework and the stability of CS is hard to deal with for a large N. We are thus motivated to reduce the stability analysis of CS on multiplex networks to a lower-dimensional problem by developing some approximation methods.

In this work, we study the coupled identical chaotic oscillators on duplex networks, specific multiplex networks with only two layers. We develop two approximation methods to deal with the stability of CS on duplex networks. In the first method, we assume that all transverse modes to synchronous chaos have the same contribution to the stability of CS. Then we obtain a modified MSF similar to the MSF on monolayer networks. In the second approx-

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imation, we only consider the contributions from a few most unstable transverse modes of layers, which are responsible for the desynchronization on each isolated layer, to the stability of CS. The stability diagrams of CS on duplex networks produced by the two approximation methods are compared with the results acquired by calculating the synchronization error in coupled chaotic oscillators. We find that the second approximation method provides better prediction on the stability of CS when N is large.

2. The model

We consider N oscillators whose time evolutions are governed by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) + \sum_{j=1}^{N} [\varepsilon^{(1)} L^{(1)}_{i,j} \boldsymbol{H}^{(1)}(\boldsymbol{x}_j) + \varepsilon^{(2)} L^{(2)}_{i,j} \boldsymbol{H}^{(2)}(\boldsymbol{x}_j)], \quad (1)$$

where $\boldsymbol{x}_i$ is the m-dimensional state variable of oscillator i, $\boldsymbol{F}(\boldsymbol{x})$ is the dynamics of an individual oscillator. These oscillators interact with each other through two different channels, each representing a different layer. To be specific, in layer 1 (or layer 2), oscillators interact with each other through the linear CF $H^{(1)}$ (or $H^{(2)}$), determining the output signal from a node on layer 1 (or layer 2), with the coupling strength $\varepsilon^{(1)}$ (or $\varepsilon^{(2)}$). $L^{(1)}$ (or $L^{(2)}$) is the Laplacian matrix characterizing the topology of layer 1 (or layer 2), with elements $L^{(1)}_{i,i} = -k^{(1)}_i$ (or $L^{(2)}_{i,i} = -k^{(2)}_i$), the degree of node i on layer 1 (or layer 2), $L^{(1)}_{i,j} = 1$ (or $L^{(2)}_{i,j} = 1$) if node i and node j are connected with a link on layer 1 (or layer 2), and $L^{(1)}_{i,j} = 0$ (or $L^{(2)}_{i,j} = 0$) otherwise. In other words, we are considering N oscillators sitting on a duplex network.

We briefly review the MSF method on the stability analysis of CS on monolayer networks

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) + \varepsilon \sum_{j=1}^{N} L_{i,j} \boldsymbol{H}(\boldsymbol{x}_j) \quad (2)$$

The variational equations of Eq. (2) with respect to CS, ($\boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N = \boldsymbol{s}$), are diagonalized into N decoupled eigenmodes of the form

$$\dot{\boldsymbol{\eta}}_i = [\mathcal{D}\boldsymbol{F}(\boldsymbol{s}) + \varepsilon\lambda_i \mathcal{D}\boldsymbol{H}(\boldsymbol{s})]\boldsymbol{\eta}_i \quad (3)$$

where λ_i ($i = 1, 2, \ldots, N$) are eigenvalues of the Laplacian matrix L and $L\phi_i = \lambda_i \phi_i$. For an undirected network, where L is symmetric, λ_i are real and can be sorted in descending order, i.e., $0 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_N$. The eigenmode ϕ_1 with $\lambda_1 = 0$ accounts for the synchronous mode and other ϕ_i ($i = 2, 3, \ldots, N$) are transverse modes. $\mathcal{D}\boldsymbol{F}(\boldsymbol{s})$ and $\mathcal{D}\boldsymbol{H}(\boldsymbol{s})$ are the $m \times m$ Jacobian matrices of the corresponding vector functions evaluated at CS. Letting $\sigma = \varepsilon\lambda$, the largest Lypunov exponent (LLE) $\Lambda(\sigma)$, determined by Eq. (3), is the so-called MSF. Generally, $\Lambda(\sigma)$ is negative when $\sigma_1 < \sigma < \sigma_2$. CS is stable when all transverse eigenmodes with $i > 1$ have negative LLE, which requires, $\sigma_1 < \varepsilon\lambda_i < \sigma_2$ for any $i > 1$. For a given node dynamics, CF can be categorized according to σ_1 and σ_2. There are three types of CF [3,39]. For type-i CFs, $\sigma_1 = \infty$ and CS is always unstable. For type-ii CFs, $\sigma_2 = \infty$ and CS is stable provided that $\varepsilon > \sigma/\lambda_2$. For type-iii CFs, both σ_1 and σ_2 are finite and CS is stable when $\sigma_1/\lambda_2 < \varepsilon < \sigma_2/\lambda_N$.

In the following, we take chaotic Lorenz oscillator ($m = 3$) as the node dynamics, which is described as $\boldsymbol{F}(\boldsymbol{x}) = [10(y - x), 28x - y - xz, xy - z]^T$. We concern with CFs whose Jacobian matrices have only one nonzero element and we denote them with their nonzero elements. Thereby, there are 9 different CFs. Fig. 1 shows the LLE, $\Lambda(\sigma)$, for 9 different CFs. The type of the CF in each plot is marked by the index in the top-right corner.

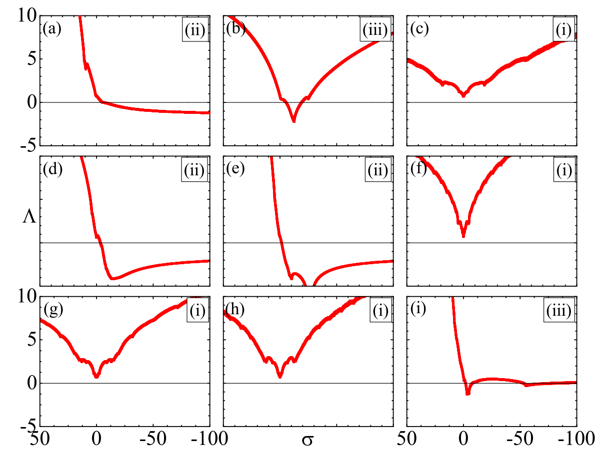

Fig. 1. The largest Lyapunov exponent Λ (in red) against σ for coupled Lorenz oscillators on monolayer networks for different CFs. (a) $\mathcal{D}\boldsymbol{H}_{1,1}$, (b) $\mathcal{D}\boldsymbol{H}_{1,2}$, (c) $\mathcal{D}\boldsymbol{H}_{1,3}$, (d) $\mathcal{D}\boldsymbol{H}_{2,1}$, (e) $\mathcal{D}\boldsymbol{H}_{2,2}$, (f) $\mathcal{D}\boldsymbol{H}_{2,3}$, (g) $\mathcal{D}\boldsymbol{H}_{3,1}$, (h) $\mathcal{D}\boldsymbol{H}_{3,2}$, and (i) $\mathcal{D}\boldsymbol{H}_{3,3}$. The index in the top-right corner in each plot denotes the type of the corresponding CF. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Numerical simulations

In this section, we numerically investigate the dependence of CS on CFs in coupled Lorenz oscillators on duplex networks with $N = 6$. Both layers of duplex networks are modeled by random networks. We have tried different realizations of duplex networks and found qualitatively similar results. Without loss of generality, we consider a specific duplex network with the Laplacians,
$$L^{(1)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 1 & 1 & -4 \end{pmatrix}$$
and
$$L^{(2)} = \begin{pmatrix} -3 & 0 & 1 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & 0 & 1 & -4 \end{pmatrix}.$$
We consider the synchronization error, which is defined as

$$\Delta = \frac{2}{N(N-1)} \sum_{i=1, j>i}^{N} \langle ||\vec{x}_j - \vec{x}_i||_2 \rangle_t, \quad (4)$$

where $\langle \cdot \rangle_t$ means the time average and $||\vec{x}_j - \vec{x}_i||_2$ is the Euclidean norm ($||\vec{x}_j - \vec{x}_i||_2 = [(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2]^{1/2}$). In each simulation, the synchronization error is averaged over 400 time units after a transient time around 2000 time units. When $\Delta < 10^{-6}$, we say that the coupled Lorenz oscillators are completely synchronized.

Fig. 1 shows the existence of all three types of CFs for Lorenz oscillators on monolayer networks. Consider coupled Lorenz oscillators on duplex networks in which two layers may take different types of CFs. There are 9 typical combinations of CFs. Then we numerically explore stable CS on the plane of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ and investigate effects of combinations of different types of CFs on CS. The results are presented in Fig. 2, where the combinations of CFs are labeled in each plot, for example (i–iii) indicating a type-i CF on layer 1 and a type-ii CF on layer 2.

The results can be summarized as follows. First, the presence of the type-i CFs always disfavors CS. When both layers take type-i CFs, it is impossible to realize CS [see Fig. 2(a)]. As shown in