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A (2+1)-dimensional breaking soliton equation: Solutions and conservation laws

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ABSTRACT

In this paper, we consider a (2+1)-dimensional breaking soliton equation which describe the (2+1)-dimensional interaction of the Riemann wave propagating along the y-axis with a long wave along the x-axis. By the Lie group analysis, the Lie point symmetry generators and symmetry reductions were deduced. From the viewpoint of exact solutions, we have performed two distinct methods to the equation for getting some exact solutions. Kudryashov's simplest methods and ansatz method with the assistance of Maple were carried out. The local conservation laws are also constructed by multiplier/homotopy methods. Finally, the graphical simulations of the exact solutions are depicted.

1. Introduction

It is well known that the nonlinear evolution equations (NLEEs) model many real world problems which comprise dispersion, convection, diffusion and nonlinear effects. If one achieve to solve the considered NLEEs via analytical or numerical techniques, then understanding the physical phenomenas of behind the model is possible.

The Korteweg–de Vries (KdV) equation which is a nonlinear and dispersive partial differential equation (PDE)

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

is a mathematically modelling the special waves which is called solitons on shallow water surfaces [1]. Solitons are localized wave disturbances that propagate without changing shape or spreading out [2]. The KdV equation has several connections to physical problems such as shallow-water waves with weakly non-linear restoring forces, long internal waves in a density-stratified ocean, ion acoustic waves in a plasma, acoustic waves on a crystal lattice [3].

After the inverse scattering method which is developed for solving the KdV equation many important approaches was presented by the researchers. In this regard, we observe many methods developed in the literature. For getting analytical solutions of considered system, Hirota bilinear method, homogeneous balance method, exp function method, ansatz method, Kudryashov simplest equation method and Lie group method can be mentioned [4–11].

For constructing the numerical solutions, variational iteration method, collocation and Galerkine spline methods can be efficient [12–14]. However, we emphasize that aforementioned all methods have some restrictions.

In this work, we study a (2+1)-dimensional breaking soliton equation [15]

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0 \tag{2}$$

This equation was used to describe the (2+1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis [15]. A class of overturning soliton solutions has been introduced in [15]. For $y = x$, and by integrating the resulting equation in (2), the equation is reduced to the KdV equation. As stated in [16], the main characteristic feature of the family of these equations is that the spectral parameter which used in the Lax representations possesses so-called breaking behaviour. It means that, the spectral value becomes a multivalued function. As a result, the solution of these equations may also become multivalued. The recent studies related with Eq. (2) is briefly its integrability and solutions. Eq. (2) was proved in [15] to be completely integrable equation. To solve Eq. (2) authors in [17] used of N-soliton solution. In [18], the authors modify the idea of three-wave method to obtain some analytic solutions by using bilinear closed form for it.

The main objectives of the under investigated paper are to study of Eq. (2) via Lie group analysis and to deduce some physical important solutions that have not reported in the literature. In

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addition, constructing the conservation laws is aimed. As known, conservation laws can be considered as sign of integrability.

The paper structured thusly. In Section 2, Lie point symmetry generators and symmetry reductions corresponding to some Lie point generators were obtained. Section 3 and 4 devoted to get exact solutions of the considered model. In this regard, Kudryashov's simplest equation method and ansatz method were employed. Section 5 is related with local conserved vectors of Eq. (2). For this goal multiplier/homotopy method were employed. Concluding remarks are given in Section 6.

2. Lie symmetry analysis and symmetry reductions

The symmetry generator [19,22,23]

$$\mathbf{X}=\xi^t(t,x,y,u)\frac{\partial}{\partial t}+\xi^x(t,x,y,u)\frac{\partial}{\partial x}+\xi^y(t,x,y,u)\frac{\partial}{\partial y}+\eta(t,x,y,u)\frac{\partial}{\partial u} \quad (3)$$

is a Lie point type symmetry of the Eq. (2) if

$$\mathbf{X}^{[4]}(u_{xt}-4u_{xy}u_x-2u_{xx}u_y-u_{xxxy})\big|_{(2)}=0, \quad (4)$$

where $\mathbf{X}^{[4]}$ is the fourth prolongation of (3). Expanding (4) and splitting on the derivatives of u leads to the following overdetermined system :

$$\xi^y_{t,t}=0,\ \ \eta_{u,t}=0,\ \ \eta_{u,u}=0,$$
$$\xi^t_x=0,\ \ \xi^x_x=-\eta_u,\ \ \xi^y_x=0$$
$$\eta_x=-\frac{\xi^y_t}{4},\ \ \xi^t_t=-2\eta_u,\ \ \xi^x_t=0,$$
$$\xi^t_u=0,\ \ \xi^x_u=0,\ \ \xi^y_u=0 \quad (5)$$

Solving the above equations we obtain the values of ξ^t, ξ^x, ξ^y and η

$$\eta=-\frac{c_1x}{4}+c_3u+f(t),$$
$$\xi^t=-2c_3t+c_4,$$
$$\xi^x=-c_3x+c_5,$$
$$\xi^y=c_1t+c_2 \quad (6)$$

with $f(t)$ being arbitrary function of t and c_1, c_2, c_3, c_4, c_5 arbitrary constants. As a result we obtain the 6-dimensional Lie algebra spanned by the following six symmetry generators:

$$\mathbf{X}_1=\frac{\partial}{\partial x},$$
$$\mathbf{X}_2=\frac{\partial}{\partial y},$$
$$\mathbf{X}_3=\frac{\partial}{\partial t},$$
$$\mathbf{X}_4=f(t)\frac{\partial}{\partial u},$$
$$\mathbf{X}_5=t\frac{\partial}{\partial y}-\frac{x}{4}\frac{\partial}{\partial u},$$
$$\mathbf{X}_6=-2t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u} \quad (7)$$

2.1. Symmetry reductions of (2)

We will consider only the linear combination of the translation symmetries, $\mathbf{\Gamma}=a_1\mathbf{X}_1+a_2\mathbf{X}_2+a_3\mathbf{X}_3$, where a_1, a_2, a_3 are constants reduces (2) to a PDE in two independent variables. The symmetry $\mathbf{\Gamma}$ yields the following three invariants:

$$f=a_2x-a_1y,\quad g=a_3x-a_1t,\quad u=\theta \quad (8)$$

Using the above invariants we can express the transformed equation as

$$-a_1a_2\theta_{fg}-a_1a_3\theta_{gg}+6a_1a_2^2\theta_f\theta_{ff}+8a_1a_2a_3\theta_f\theta_{fg}+2a_1a_3^2\theta_f\theta_{gg}+4a_1a_3^2\theta_g\theta_{fg}$$
$$+4a_1a_2a_3\theta_g\theta_{ff}+a_1a_2^3\theta_{ffff}+3a_1a_2^2a_3\theta_{fffg}+3a_1a_2a_3^2\theta_{ffgg}+a_1a_3^3\theta_{fggg}=0 \quad (9)$$

which is a nonlinear PDE in two independent variables f and g. We can further reduce this equation using its symmetries. The vector field

$$\mathbf{Y}=\xi^f(f,g,\theta)\frac{\partial}{\partial f}+\xi^g(f,g,\theta)\frac{\partial}{\partial g}+\eta(f,g,\theta)\frac{\partial}{\partial \theta} \quad (10)$$

is a Lie point symmetry of the Eq. (9) if

$$\mathbf{Y}^{[4]}(-a_1a_2\theta_{fg}-a_1a_3\theta_{gg}+6a_1a_2^2\theta_f\theta_{ff}+8a_1a_2a_3\theta_f\theta_{fg}+2a_1a_3^2\theta_f\theta_{gg}+4a_1a_3^2\theta_g\theta_{fg}$$
$$+4a_1a_2a_3\theta_g\theta_{ff}+a_1a_2^3\theta_{ffff}+3a_1a_2^2a_3\theta_{fffg}+3a_1a_2a_3^2\theta_{ffgg}+a_1a_3^3\theta_{fggg})\big|_{(9)}=0, \quad (11)$$

where $\mathbf{Y}^{[4]}$ is the fourth prolongation of (10). Expanding (11) and splitting on the derivatives of θ leads to the following overdetermined system:

$$\eta_{\theta,g}=0,\quad \eta_{\theta,\theta}=0,\quad \xi^f_f=\frac{-2a_3^2\eta_g+2a_2\eta_\theta}{a_2},$$
$$\xi^g_f=0,\quad \eta_f=\frac{-2a_3^2\eta_g+a_2\eta_\theta}{2a_2a_3},$$
$$\xi^f_g=\frac{2a_3^2\eta_g-3a_2\eta_\theta}{a_3},\quad \xi^g_g=-\eta_\theta,\quad \xi^f_\theta=0,\quad \xi^g_\theta=0 \quad (12)$$

Solving the above equations we get the values of ξ^f, ξ^g and η

$$\eta=c_1\theta+\frac{c_1f}{2a_3}+h\left(\frac{a_2g-a_3f}{a_2}\right)$$
$$\xi^f=2h\left(-\frac{a_3f}{a_2}+g\right)a_3+2c_1f-\frac{3c_1a_2g}{a_3}+c_3$$
$$\xi^g=-c_1g+c_2$$

As a result we obtain the 4-dimensional Lie algebra spanned by the following four symmetries:

$$\mathbf{Y}_1=\frac{\partial}{\partial f},$$
$$\mathbf{Y}_2=\frac{\partial}{\partial g},$$
$$\mathbf{Y}_3=-\frac{(3a_2g-2a_3f)}{a_3}\frac{\partial}{\partial f}-g\frac{\partial}{\partial g}+\frac{2a_3\theta+f}{2a_3}\frac{\partial}{\partial \theta},$$
$$\mathbf{Y}_4=2h\left(\frac{a_2g-a_3f}{a_2}\right)a_3\frac{\partial}{\partial f}+h\left(\frac{a_2g-a_3f}{a_2}\right)\frac{\partial}{\partial \theta} \quad (13)$$