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Some special number sequences obtained from a difference equation of degree three

Cristina Flaut[1,*], Diana Savin[2]

Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, Constanta 900527, Romania

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ABSTRACT

In this paper we present applications of special numbers obtained from a difference equation of degree three. As a particular case of this difference equation of degree three, we obtain the generalized Pell-Fibonacci-Lucas numbers, which were extended to the generalized quaternion algebras. Using properties of these quaternion elements, we can define a set with an interesting algebraic structure, namely, an order on a generalized rational quaternion algebra. Another presented application is in the Coding Theory, since some of these numbers can be used to built cyclic codes with good properties (MDS codes).

1. Introduction

Let n be an arbitrary positive integer and let a, b, c, x_0, x_1, x_2 be arbitrary integers. We consider the following difference equation of degree three

$$D_n = aD_{n-1} + bD_{n-2} + cD_{n-3}, D_0 = x_0, D_1 = x_1, D_2 = x_2. \tag{1.1}$$

If we consider $a = b = 1, c = 0,\ x_0 = 0, x_1 = 1, x_2 = 1,$ we obtain the Fibonacci numbers and if we take $a = b = 1, c = 0, x_0 = 2, x_1 = 1, x_2 = 3,$ we get the Lucas numbers. In the same way, for $a = 1, b = 0, c = 1, x_0 = 0, x_1 = 1, x_2 = 1,$ we find the Fibonacci-Narayana numbers [9].

If we take π a positive prime integer and we consider the sequence given in (1.1), we obtain a difference equation of degree three, defined on $\mathbb{Z}_\pi$,where $\mathbb{Z}_\pi$ denotes the finite field of integers modulo π. The numbers obtained in this case are repeated after a certain period. We denote with $l_d(\pi)$ and $\beta_d(\pi)$ the period, respectively the number of zeros in a single period for this sequence.

Some properties of the above numbers were studied in various papers. Some of them are: [1,2,4–7,11–13,16,17,20,21,23]. In this paper, in Section 2, we extend the numbers given by relation (1.1) to generalized quaternions, obtaining an interesting algebraic structure. This algebraically structure can't be extended to Octonion algebras or to other algebras obtained by the Cayley–Dickson process of dimension greater than 16, since these algebras are not associative. In these cases, we can obtain only a weak structure: a module or a vector space. In Section 3, an application in Coding Theory is given. These numbers, arising from the same Eq. (1.1) but defined on $\mathbb{Z}_\pi$, were used to built cyclic codes with good properties (MDS codes).

2. Some applications of generalized Pell-Fibonacci-Lucas elements

In relation (1.1), if we consider $a = 2, b = 1, c = 0, x_0 = 0, x_1 = 1, x_2 = 2,$ we obtain the Pell numbers and if we take $a = 2, b = 1, c = 0, x_0 = 2, x_1 = 2, x_2 = 6,$ we obtain the Pell-Lucas numbers. Let $(P_n)_{n\geq 0}$ be the Pell sequence

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2, P_0 = 0; P_1 = 1,$$

and $(Q_n)_{n\geq 0}$ be the Pell-Lucas sequence

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2, Q_0 = 2; Q_1 = 2.$$

* Corresponding author.
E-mail addresses: cflaut@univ-ovidius.ro, cristina_flaut@yahoo.com (C. Flaut), savin.diana@univ-ovidius.ro, dianet72@yahoo.com (D. Savin).
[1] http://cristinaflaut.wikispaces.com/; http://www.univ-ovidius.ro/math/.
[2] http://www.univ-ovidius.ro/math/.

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We consider the numbers $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. The following formulae are well known:

Binet's formula for Pell sequence

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad \text{for all } n \in \mathbb{N};$$

Binet's formula for Pell-Lucas sequence

$$Q_n = \alpha^n + \beta^n, \text{ for all } \; n \in \mathbb{N}.$$

Let A be the generating function of the sequence $(D_n)_{n\geq 0}$, $A(z) = \sum_{n\geq 0} D_n z^n$. In the following, we determine this function.

Proposition 2.1. *We have:*

$$A(z) = \frac{D_0 + (D_1 - aD_0)z + (D_2 - aD_1 - bD_0)z^2}{1 - az - bz^2 - cz^3}.$$

Proof.

$$A(z) = D_0 + D_1 z + D_2 z^2 + D_3 z^3 + \ldots + D_n z^n + \ldots \tag{2.1}$$

$$azA(z) = aD_0 z + aD_1 z^2 + aD_2 z^3 + \ldots + aD_{n-1} z^n + \ldots \tag{2.2}$$

$$bz^2 A(z) = bD_0 z^2 + bD_1 z^3 + bD_2 z^4 + \ldots + bD_{n-2} z^n + \ldots \tag{2.3}$$

$$cz^3 A(z) = cD_0 z^3 + cD_1 z^4 + cD_2 z^5 + \ldots + cD_{n-3} z^n + \ldots \tag{2.4}$$

Adding the equalities (2.2)–(2.4) and substracting relation (2.1), we obtain:

$$A(z)\left(1 - az - bz^2 - cz^3\right) = D_0 + (D_1 - aD_0)z + (D_2 - aD_1 - bD_0)z^2.$$

Therefore, we get

$$A(z) = \frac{D_0 + (D_1 - aD_0)z + (D_2 - aD_1 - bD_0)z^2}{1 - az - bz^2 - cz^3}.$$

□

We remark that for $a = c = 1$ and $b = 3$, we obtain the sequence $(D_n)_{n\geq 0}$, $D_0 = 0$, $D_1 = D_2 = 1$, $D_{n+3} = D_{n+2} + 3D_{n+1} + D_n$, $n \geq 0$. This equality is equivalent with

$$D_{n+3} + D_{n+2} = 2(D_{n+2} + D_{n+1}) + (D_{n+1} + D_n).$$

If we take the sequence $(b_n)_{n\geq 0}$, $b_{n+1} = D_{n+1} + D_n$, $n \geq 0$, then the last relation becomes

$$b_{n+3} = 2b_{n+2} + b_{n+1}, \; n \geq 0,$$

where $b_1 = 1$ and $b_2 = 2$. Moreover, if we consider $b_0 = 0$, then it results that the sequence $(b_n)_{n\geq 0}$ is the sequence of Pell numbers $(P_n)_{n\geq 0}$.

Proposition 2.2. *Let $(P_n)_{n\geq 0}$ be the sequence of Pell numbers and $(Q_n)_{n\geq 0}$ be the sequence of Pell-Lucas numbers. Let A be the following matrix* $A = \begin{pmatrix} \frac{Q_1}{2} & 0 & \sqrt{2}P_1 \\ 0 & \frac{Q_1}{2} + \sqrt{2}P_1 & 0 \\ \sqrt{2}P_1 & 0 & \frac{Q_1}{2} \end{pmatrix}$. *It results that*

$$A^n = \begin{pmatrix} \frac{Q_n}{2} & 0 & \sqrt{2}P_n \\ 0 & \frac{Q_n}{2} + \sqrt{2}P_n & 0 \\ \sqrt{2}P_n & 0 & \frac{Q_n}{2} \end{pmatrix}.$$

Proof. We prove the following statement, by induction after $n \in \mathbb{N}^*$

$$P(n) : A^n = \begin{pmatrix} \frac{Q_n}{2} & 0 & \sqrt{2}P_n \\ 0 & \frac{Q_n}{2} + \sqrt{2}P_n & 0 \\ \sqrt{2}P_n & 0 & \frac{Q_n}{2} \end{pmatrix}.$$

We remark that $P(1)$ is true.

Assuming that $P(n)$ is true, we prove that $P(n+1)$ is true.

$$P(n+1) : A^{n+1} = \begin{pmatrix} \frac{Q_{n+1}}{2} & 0 & \sqrt{2}P_{n+1} \\ 0 & \frac{Q_{n+1}}{2} + \sqrt{2}P_{n+1} & 0 \\ \sqrt{2}P_{n+1} & 0 & \frac{Q_{n+1}}{2} \end{pmatrix}.$$

$$A^{n+1} = A^n A = \begin{pmatrix} \frac{Q_n}{2} & 0 & \sqrt{2}P_n \\ 0 & \frac{Q_n}{2} + \sqrt{2}P_n & 0 \\ \sqrt{2}P_n & 0 & \frac{Q_n}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{Q_1}{2} & 0 & \sqrt{2}P_1 \\ 0 & \frac{Q_1}{2} + \sqrt{2}P_1 & 0 \\ \sqrt{2}P_1 & 0 & \frac{Q_1}{2} \end{pmatrix}.$$

Using Binet's formulae for Pell-Lucas sequence, we obtain:

$$A^{n+1} = \begin{pmatrix} \frac{\alpha^n + \beta^n}{2} & 0 & \frac{\alpha^n - \beta^n}{2} \\ 0 & \alpha^n & 0 \\ \frac{\alpha^n - \beta^n}{2} & 0 & \frac{\alpha^n + \beta^n}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\alpha + \beta}{2} & 0 & \frac{\alpha - \beta}{2} \\ 0 & \alpha & 0 \\ \frac{\alpha - \beta}{2} & 0 & \frac{\alpha + \beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha^{n+1} + \beta^{n+1}}{2} & 0 & \frac{\alpha^{n+1} - \beta^{n+1}}{2} \\ 0 & \alpha^{n+1} & 0 \\ \frac{\alpha^{n+1} - \beta^{n+1}}{2} & 0 & \frac{\alpha^{n+1} + \beta^{n+1}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{Q_{n+1}}{2} & 0 & \sqrt{2}P_{n+1} \\ 0 & \frac{Q_{n+1}}{2} + \sqrt{2}P_{n+1} & 0 \\ \sqrt{2}P_{n+1} & 0 & \frac{Q_{n+1}}{2} \end{pmatrix}.$$

Therefore, $P(n+1)$ is true. □

In the papers [8,18] were introduced the generalized Fibonacci-Lucas numbers and the generalized Fibonacci-Lucas quaternions and were obtained many properties of these quaternion elements. In a similar way, we introduce here the generalized Pell-Fibonacci-Lucas numbers and the generalized Pell-Fibonacci-Lucas quaternions.

First of all, we give some identities involving Pell numbers and Pell-Lucas numbers.

Proposition 2.3. *Let $(P_n)_{n\geq 0}$ be the sequence of Pell numbers and $(Q_n)_{n\geq 0}$ be the sequence of Pell-Lucas numbers [10]. The following statements are true:*

(i)

$$Q_n Q_{n+l} = Q_{2n+l} + (-1)^n Q_l, \; n, l \in \mathbb{N};$$