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Doubling metric Diophantine approximation in the dynamical system of continued fractions

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ABSTRACT

This paper is concerned with the Diophantine properties of the orbits of real numbers in continued fraction system under the doubling metric. More precisely, let φ be a positive function defined on $\mathbb{N}$. We determine the Lebesgue measure and Hausdorff dimension of the set

$$E(\varphi) = \{(x, y) \in [0, 1) \times [0, 1) : |T^n x - y| < \varphi(n) \text{ for i.m. } n\},$$

where T is the Gauss map and "i.m." stands for "infinitely many".

1. Introduction

Each irrational number x admits a unique infinite continued fraction expansion of the form:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}} = [a_1(x), a_2(x), \cdots] \tag{1.1}$$

where $a_n(x) \in \mathbb{N}$ are called the partial quotients of x.

This can be generated by using the Gauss transformation T: [0, 1)→ [0, 1) defined by

$$Tx = \begin{cases} 0, & x = 0, \\ \{\frac{1}{x}\}, & x \in (0, 1). \end{cases}$$

where {} denotes the fractional part.

As is well known, the Gauss transformation is ergodic with respect to the Gauss measure μ given by $d\mu = \frac{dx}{(1+x)\log 2}$ which is equivalent to the Lebesgue measure $\mathcal{L}$. Birkhoff's ergodic theorem implies that for any ball $B \subseteq [0, 1)$ with positive radius, the set

$$E := \{x \in [0, 1) : T^n x \in B \text{ for i.m. } n\}$$

has full measure. So, for a further study, one would like to ask what will happen if the ball B shrinks with the time.

Given a sequence of balls $\{B(y_0, \varphi(n)\}$ with fixed center $y_0 \in [0, 1)$ and decreasing radius $\{\varphi(n)\}_{n \ge 1}$, define

$$E(\varphi, y_0) := \{x \in [0, 1) : |T^n x - y_0| < \varphi(n) \text{ for i.m. } n\}.$$

One may characterize the size of the set $E(\varphi, y_0)$ in the sense of Lebesgue measure and Hausdorff dimension. This is called shrinking target problem initialed by Hill and Velani [5]. Philipp [10] and Li et al.[7] determined the Lebesgue measure and Hausdorff dimension of the set $E(\varphi, y_0)$ respectively.

All these results concern the case when y_0 is fixed. So how about the case when y_0 is also involved in. In other words, we consider the following doubling metric shrinking target problem. Let φ be a positive function defined on $\mathbb{N}$. We determine the size of the set

$$E(\varphi) := \{(x, y) \in [0, 1) \times [0, 1) : |T^n x - y| < \varphi(n) \text{ for i.m. } n\}.$$

The results are as follows.

Theorem 1.1. *Let φ be a positive function defined on $\mathbb{N}$. Then*

$$\mathcal{L}^2(E(\varphi)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \varphi(n) < \infty, \\ 1, & \text{if } \sum_{n=1}^{\infty} \varphi(n) = \infty, \end{cases}$$

where $\mathcal{L}^2$ denotes the two dimensional Lebesgue measure.

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Theorem 1.2. *Let φ be a positive function defined on $\mathbb{N}$. Then*

$$\dim_H(E(\varphi)) = \begin{cases} 2, \quad \text{when } B = 1, \\ 1 + \inf\{s \geq 0 : P(-s(\log|T'| + \log B)) \leq 0\}, \\ \quad \text{when } B \in (1, \infty), \\ 1 + \frac{1}{1+b}, \quad \text{when } B = \infty, \end{cases}$$

where $P(\cdot)$ denotes the pressure function, $\log B = \liminf\limits_{n\to\infty} \frac{-\log\varphi(n)}{n}$ and $\log b = \liminf\limits_{n\to\infty} \frac{\log(-\log\varphi(n))}{n}$.

For the doubling metric properties, Dodson [1] considered the question in the classic Diophantine approximation, namely the set

$\{(x, y) \in [0, 1) \times [0, 1) : ||nx - y|| < \phi(n) \text{ for i.m. } n\}$.

This setting was transformed to the shrinking target problem by Ge and Lü [4] in the system of $\beta-$transformation. In this paper, what we do is to consider this doubling metric properties in continued fraction transformation.

2. Preliminary

This section is devoted to collecting some basic properties of continued fractions and some metric results in continued fractions.

Call the finite truncation of (1.1), $p_n/q_n := [a_1(x),\ a_2(x),\ \cdots,\ a_n(x)]$, the convergent of x. It is known that p_n, q_n are decided by the following recursive relations:

$p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2},$

under the conventions $p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1$.

For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, we call

$I_n(a_1, a_2, \cdots, a_n) = \{x \in [0, 1) : a_i(x) = a_i \text{ for all } 1 \leq i \leq n\}$

a cylinder of order n.

The first lemma collects some basic properties of continued fractions needed later.

Lemma 2.1 [6]**.** *Let $a_1, a_2, \cdots, a_n \in \mathbb{N}$. We have*

(i) $q_n \geq 2^{(n-1)/2}$, $\prod\limits_{i=1}^{n} a_i \leq q_n \leq 2^n \prod\limits_{i=1}^{n} a_i$, $|q_n x - p_n| \leq \frac{1}{q_{n+1}}$;

(ii) $[0, 1) = \bigcup\limits_{(a_1,a_2,\cdots,a_n)\in\mathbb{N}^n} I_n(a_1, a_2, \cdots, a_n)$ and $|I_n(a_1, a_2, \cdots, a_n)| = \frac{1}{q_n(q_n+q_{n-1})}$, *where $|D|$ denote the diameter of the set D;*

(iii) *for any $x \in I_n(a_1,\ a_2,\ \cdots,\ a_n)$, $T^n x = -\frac{q_n x - p_n}{q_{n-1}x - p_{n-1}}$* .

Let φ be a positive function defined on $\mathbb{N}$. Define

$E(\varphi, y_0) = \{x \in [0, 1) : |T^n x - y_0| < \varphi(n) \text{for i.m.} n\}$.

In the sense of Lebesgue measure, Philipp proved that the following dynamical Borel–Cantelli lemma for Gauss transformation.

Lemma 2.2 [10]**.**

$$\mathcal{L}(E(\varphi, y_0)) = \begin{cases} 0, \quad \text{if } \sum\limits_{n=1}^{\infty} \varphi(n) < \infty, \\ 1, \quad \text{if } \sum\limits_{n=1}^{\infty} \varphi(n) = \infty. \end{cases}$$

In the sense of dimension, Li–Wang–Wu–Xu showed that

Lemma 2.3 [7]**.**

$$\dim_H((E(\varphi, y_0))) = \begin{cases} 1, \quad \text{when } B = 1, \\ \inf\{s \geq 0 : P(-s(\log|T'| + \log B)) \leq 0\}, \\ \quad \text{when } B \in (1, \infty), \\ \frac{1}{1+b}, \quad \text{when } B = \infty, \end{cases}$$

where $P(\cdot)$ denotes the pressure function, $\log B = \liminf\limits_{n\to\infty} \frac{-\log\varphi(n)}{n}$ and $\log b = \liminf\limits_{n\to\infty} \frac{\log(-\log\varphi(n))}{n}$.

Next we recall some results about pressure function in continued fractions. Let $\phi : [0, 1) \to \mathbb{R}$. The pressure function P with respect to the potential ϕ is defined by

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,a_2,\cdots,a_n)\in\mathbb{N}^n} \sup_{x\in[0,1)} e^{S_n\phi[a_1,a_2,\cdots,a_n+x]},$$

where $S_n\phi(x)$ denotes the ergodic sum $\phi(x) + \phi(Tx) + \cdots + \phi(T^{n-1}x)$.

Let $B > 1$ be a fixed number and $\Theta_n(s) = \sum\limits_{(a_1,a_2,\cdots,a_n)\in\mathbb{N}^n} \frac{1}{(B^n q_n^2)^s}$.

Define

$s_{n,B} := \inf\{s \geq 0 : \Theta_n(s) \leq 1\},$

$s_B := \inf\{s \geq 0 : P(-s(\log|T'| + \log B)) \leq 0\}.$

Lemma 2.4 [9,11]**.** $\lim\limits_{n\to\infty} s_{n,B} = s_B$. *What's more,* $\lim\limits_{B\to\infty} s_B = \frac{1}{2}$, $\lim\limits_{B\to 1} s_B = 1$ *and s_B is continuous with respect to $B \in (1, \infty)$.*

Falconer's slicing lemma and product formula are cited here for later use.

Lemma 2.5 [2]**.** *Let F be any subset of $\mathbb{R}^2$, and let E be a subset of the $x-$axis. L_x denotes the line parallel to the $y-$axis through the point $(x, 0)$. If $\dim_H(F \bigcap L_x) \geq t$ for all $x \in E$, then $\dim_H F \geq t + \dim_H E$.*

Lemma 2.6 [2]**.** *For any sets $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^n$*

$\dim_H(E \times F) \leq \dim_H E + \overline{\dim}_B F,$

where $\overline{\dim}_B$ denotes the boxing counting dimension.

A dimensional result is also needed later, so we cite here.

Lemma 2.7 [3,8]**.** *For any c, $b > 1$,*

$$\dim_H\{x \in [0, 1) : a_n(x) \geq c^{b^n} \text{ for i.m. } n\} = \frac{1}{1+b}.$$

According to Lemmas 2.7 and 2.1 (i) we can easily get:

Corollary 2.8. *For any c, $b > 1$, $\dim_H\{x \in [0, 1) : q_n(x) \geq c^{b^n}$ for i.m. $n\} = \frac{1}{1+b}$.*

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For any fixed point y, let

$E_y(\varphi) = \{x \in [0, 1) : |T^n x - y| < \varphi(n) \text{for i.m.} n\}$.

Then by Lemma 2.2, we get

$$\mathcal{L}(E_y(\varphi)) = \begin{cases} 0, \quad \text{if } \sum\limits_{n=1}^{\infty} \varphi(n) < \infty, \\ 1, \quad \text{if } \sum\limits_{n=1}^{\infty} \varphi(n) = \infty. \end{cases}$$

We can calculate the Lebesgue measure of $E(\varphi)$ by applying Fubini's theorem. More precisely,

$$\begin{aligned} \mathcal{L}^2(E(\varphi)) &= \int_0^1 \int_0^1 \chi_{E(\varphi)} dxdy \\ &= \int_0^1 (\int_0^1 \chi_{E_y(\varphi)} dx) dy = \int_0^1 \mathcal{L}(E_y(\varphi)) dy, \end{aligned}$$

where χ_E denotes the indicator function of the set E.