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# Smooth Quintic spline approximation for nonlinear Schrödinger equations with variable coefficients in one and two dimensions

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ABSTRACT

The present paper uses a relatively new approach and methodology to solve one and two dimensional nonlinear Schrödinger equations numerically. We use the horizontal method of lines and $\theta$-method, $\theta \in [1/2, 1]$ for time discretization that reduces the problem into an amenable system of ordinary differential equations. The resulting system of ODEs in space subsequently have been solved by quintic polynomial spline scheme. Convergence of the scheme in maximum norm is established rigorously. The convergence orders are $\mathcal{O}(k + h_x^4 + h_y^4)$ and $\mathcal{O}(k^2 + h_x^4 + h_y^4)$, where $k$ is the temporal grid size and $h_x$ and $h_y$ are spatial grid sizes, respectively. Matrix stability analysis shows that the method is conditionally stable. The efficacy of proposed approach has been confirmed with four numerical experiments, where comparison is made with some earlier works. It is clear that the results obtained are acceptable and are in good agreement with earlier studies. The present scheme is very simple, effective and convenient for obtaining numerical solution of Schrödinger equation.



## 1. Introduction

In mathematics and physics, nonlinear partial differential equations are partial differential equations with nonlinear terms. The nonlinear Schrödinger equation appears more and more frequently in many mathematical physics applications. Various physical phenomena in the fields of hydrodynamics, nonlinear optic, self-focusing in laser pulses, thermodynamic processes in meso scale systems, propagation of heat pulses in crystals, helical motion of very thin vortex filaments, models of protein dynamics, magnetic thin films, description of the dynamics of Bose-Einstein condensate at extremely low temperature, models of energy transfer in molecular systems and plasma are successfully described by nonlinear Schrödinger equations (see [1–17]).

In this article, we will develop an approximation based on exponential spline to obtain numerical solution of the following generalized nonlinear Schrödinger equation with variable coefficients:

$$\imath\frac{\partial u}{\partial t}+A(x,y,t)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+B(x,y,t)u+C(x,y,t)\Psi(|u|^2)u=0, \tag{1.1}$$

$$(x,y,t)\in\Omega\times[0,T],\quad \Omega\subset\mathbb{R}^d,\quad d=1,2,$$

with the initial condition

$$u(x,y,0)=\phi(x,y),\quad (x,y,z)\in\Omega, \tag{1.2}$$

and the boundary conditions

$$u(x,y,t)=f(x,y,t),\quad (x,y,t)\in\partial\Omega,\quad t\in(0,T], \tag{1.3}$$

where $\Omega=[L_x,R_x]\times[L_y,R_y]$ is a two-dimensional rectangle domain, $(0, T]$ is the time interval, $\imath=\sqrt{-1}$ is the complex unit, $A(x, y, t)$ and $B(x, y, t)$ are bounded real functions, $\Psi$ is a given real-valued function, $\partial\Omega$ is the boundary of $\Omega$, $\phi$, $f$ are given sufficiently smooth functions, $C(x, y, t)$ is the arbitrary real-valued potential function, $u(x, y, t)$ is an unknown complex-valued wave function which describes the motion of soliton(s) [18]. When $\Psi = 0$, Eq. (1.1) is the linear Schrödinger equation, which is studied extensively in the literature [19–28]. When $\Psi(|u|^2)=|u|^2$, then Eq. (1.1) is the cubic NLSE [18,29–38].

The nonlinear Schrödinger Eq. (1.1) conserves many quantities. Among them, the mass (or wave energy in nonlinear optics) and energy (or Hamiltonian in nonlinear optics) conservation are given as:

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$$\mathfrak{M}(t)=\int_{\mathbb{R}^d}|u(x,y,t)|^2dx=\int_{\mathbb{R}^d}|u(x,y,0)|^2dx=\mathfrak{M}(0),\qquad \forall t>0. \tag{1.4}$$

If $A(x, y, t)$, $B(x, y, t)$ and $C(x, y, t)$ are independent of $t$ (i.e. $A(x, y, t)\equiv A$ and $C(x, y, t)\equiv C$) then

$$\begin{aligned}\mathfrak{E}(t) &= \int_{\mathbb{R}^d}\Big(A|u_x(x,y,t)+u_y(x,y,t)|^2-C(x,y)|u(x,y,t)|^2\\ &\quad -\frac{C}{2}|u(x,y,t)|^4\Big)dx\\ &= \int_{\mathbb{R}^d}\Big(A|u_x(x,y,0)+u_y(x,y,0)|^2-C(x,y)|u(x,y,0)|^2\\ &\quad -\frac{C}{2}|u(x,y,0)|^4\Big)dx \\ &= \mathfrak{E}(0),\qquad t>0.\end{aligned} \tag{1.5}$$

As everyone knows, most differential equations cannot be solved analytically. Developing efficient numerical methods for solving differential equations, especially partial differential equations, is necessary and important. A great deal of research on the numerical solution of partial differential equation has been done. Due to the wide applications of the Schrödinger equations, performing efficient and accurate numerical simulations for the Schrödinger equations plays an essential role in many real applications. For the linear Schrödinger equation, there are a lot of numerical studies in the literature [19–28]. We will not describe these literature in details since we will focus on the numerical study for the generalized NLSE (1.1) in this paper. For cubic NLSE, numerical studies were also reported extensively in the literature [29–55].

For example, Dehghan and Mirzaei [22] proposed a meshless local boundary integral equation (LBIE) method to solve the unsteady two-dimensional Schrödinger equation. Dehghan and Emami–Naeini [24] illustrated the application of Sinc-collocation and Sinc–Galerkin methods to the approximate solution of the two-dimensional time dependent Schrödinger equation with non-homogeneous boundary conditions. The alternating direction implicit compact finite difference schemes are devised for the numerical solution of two-dimensional Schrödinger equations by Gao and Xie [28]. Li et al. [33] studied the one-dimensional cubic NLSE with wave operator by the compact finite difference method. Wang et al. [35] proposed a fourth-order compact and energy conservative difference scheme for the two-dimensional cubic NLSE with periodic boundary conditions. In addition, based on the standard fourth-order compact finite difference method, i.e., fourth-order Pade approximations for the second derivative, Xu and Zhang [36] proposed two unconditionally stable ADI methods with spatial fourth-order accuracy and temporal second-order accuracy to solve the two-dimensional cubic NLSE. Based on the combined compact difference scheme, an alternating direction implicit method is proposed for solving two-dimensional cubic nonlinear Schrödinger equations by Li et al. [38]. The proposed method was sixth-order accurate in space and second-order accurate in time. El-Danaf et al. [45] were concerned with the problem of applying cubic non-polynomial spline functions to develop a numerical method for obtaining approximation for the solution for cubic non-linear Schrödinger equation. Lin [46,47] presented numerical methods based on parametric cubic and septic splines for solving the cubic nonlinear Schrödinger equation. Mohammadi [48] implemented the exponential spline scheme to find a numerical solution of the nonlinear Schrödinger equations with constant and variable coefficients. Two-dimensional Schrödinger equations are solved via differential quadrature method by Golbabai and Nikpour [49]. Rongpei Zhang et al. [51] presented a conservative Fourier spectral collocation (FSC) method to solve the two-dimensional nonlinear Schrödinger (NLS) equation. An effective differential quadrature method (DQM) which is based on modified cubic B- spline (MCB) has been implemented to obtain the numerical solutions for the nonlinear Schrödinger (NLS) equation by Bashan et al. [53]. Dehghan and Mohammadi [54] applied the RBF-FD technique for the nonlinear Schrödinger equation in two and three dimensions. Zhang et al. [55–57] proposed the Ritz methods to predict numerical solutions for the two-dimensional nonlinear Schrödinger, three-dimensional wave and generalized regularized long wave equations.

The purpose of this paper is to give a new spline method that is based on a quintic polynomial spline function of the form $\sum_{i=0}^{5} c_i x^i$ to develop numerical methods for obtaining smooth approximations for the solution of the problem (1.1)–(1.3). According to Larry [58] the space $T_5 = span\{1, x, x^2, \ldots, x^5\}$ generates an extended complete Chebyshev space on $\Omega$.

This approach has its own advantages in comparison with finite difference methods. For example, once the solution has been computed, the information needed for spline interpolation between mesh points is available. This is important when the solution of the boundary value problem is required at different locations in interval $\Omega$. This approach has added advantage that it not only provides continuous approximations to $u(x, y, t)$, but also for its derivatives at every point of the range of integration. We give the truncation error of the method and convergence analysis. The analysis will be illustrated by investigating some examples. The numerical simulations validate and demonstrate the advantages of the method.

A brief outline of the remainder of the paper is as follows. In Section 2, the quintic polynomial spline formulation is derived for the numerical solution of Eq. (1.1) in space directions. Also, we derive the needed boundary formulas in this section. In Section 3, we present the formulation of our method. The convergence analysis of presented method is discussed in Section 4. Moreover in Section 5, the stability of the proposed numerical method is investigated. In Section 6, we present results of numerical experiments demonstrating the expected global accuracy of the method and its efficacy on several test problems from the literature. The paper ends with some concluding remarks and a brief discussion in Section 7.

## 2. Quintic polynomial spline functions

The solution domain, $\Omega = \{(x, y, t); L_x \le x \le R_x, L_y \le y \le R_y, t > 0\}$, is divided to $N_s \times N_s \times N_t$ mesh. The grid points are $(x_i, y_l, t_j)$, where $x_i = L_x + ih_x$; $h_x = \frac{R_x - L_x}{N_s}$, $i = 0, 1, \ldots, N_s$, $y_l = L_y + lh_y$; $h_y = \frac{R_y - L_y}{N_s}$, $l = 0, 1, \ldots, N_s$ and $t_j = jk$; $k = \frac{T}{N_t}$, $j = 0, 1, 2, \ldots, N_t$, $N_s$ and $N_t$ are positive integers.

Let $S^j_{(1)i,l}$ be an approximation to $u^j_{i,l} = u(x_i, y_l, t_j)$, obtained by the segment $S_{(1)i}(x, y_l, t_j)$ of the quintic polynomial spline functions $S_{(1)i}(x, y_l, t_j) \in C^4[L_x, R_x]$ passing through the points $(x_{i+1}, y_l, t_j)$ and $(x_i, y_l, t_j)$ and is defined by

$$S_{(1)i}(x, y_l, t_j) = \sum_{r=0}^{5} a_i^{(r)}(y_l, t_j)(x - x_i)^r, \quad i = 0, 1, \ldots, N_s, \tag{2.1}$$

where $a_i^{(0)}(y_l, t_j)$, $a_i^{(1)}(y_l, t_j)$, $a_i^{(2)}(y_l, t_j)$, $a_i^{(3)}(y_l, t_j)$, $a_i^{(4)}(y_l, t_j)$ and $a_i^{(5)}(y_l, t_j)$ are unknown coefficients.

Also, let $S^j_{(2)i,l}$ be an approximation to $u^j_{i,l} = u(x_i, y_l, t_j)$, obtained by the segment $S_{(2)l}(x_i, y, t_j)$ of the quintic polynomial spline functions $S_{(2)l}(x_i, y, t_j) \in C^4[L_y, R_y]$ passing through the points $(x_i, y_{l+1}, t_j)$ and $(x_i, y_l, t_j)$ and is defined by

$$S_{(2)l}(x_i, y, t_j) = \sum_{r=0}^{5} a_l^{*(r)}(x_i, t_j)(y - y_l)^r, \quad l = 0, 1, \ldots, N_s, \tag{2.2}$$