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Parrondo's paradox or chaos control in discrete two-dimensional dynamic systems

Steve A. Mendoza[a], Eliza W. Matt[b], Diego R. Guimarães-Blandón[a], Enrique Peacock-López[a,*]

[a] Department of Chemistry, Williams College, Williamstown, MA 01267, USA
[b] Department of Mathematics, Williams College, Williamstown, MA 01267, USA

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ABSTRACT

In ecological modeling, seasonality can be represented as an alternation between environmental conditions. This concept of alternation holds common ground between ecologists and chemists, who design time-dependent settings for chemical reactors to influence the yield of a desired product. In this study and for a variety of maps, we consider a switching strategy that alternates between two undesirable dynamics that yields a stable desirable dynamic behavior. By comparing bifurcation diagrams of a map and its alternate version, we can easily find parameter values, which, on their own, yield chaotic orbits. When alternated, however, the parameter values yield a stable periodic orbit. Our analysis of the two-dimensional (2-D) maps is an extension of our previous work with one-dimensional (1-D) maps. In the case of 2-D maps, we consider the Beddington, Free, and Lawton and Udwadia and Raju maps. For these 2-D maps, we not only show that we can find "chaotic" parameters for the so-called "chaos" + "chaos" = "periodic" case, but we find two new "desirable" dynamic situations: "quasiperiodic" + "quasiperiodic" = "periodic" and "chaos" + "chaos" = "periodic coexistence." In the former case, the alternation of chaotic dynamics yield two different periodic stable orbits implying the coexistence of attractors.

1. Introduction

Over the years, theoretical ecologists have modeled population dynamics using either discrete or continuous equation methods [1–8]. In the former case, maps have been the method of choice [1–6]. In particular, the logistic map has played a central role in the development and understanding of complex dynamic systems. The logistic map was originally used to study populations of non-overlapping generations, and is represented by the following relation between the new generation (X_{n+1}) and the old generation (X_n).

Independently of ecological studies, alternate dynamics strategies have been the center of attention of theoretical analyses due to the so-called Parrondo's paradox, where two losing games can be combined to yield a winning game [9,10]. For more details of Parrondo's paradox and applications, we refer our readers to papers by Parrondo and co-workers [10–16]. In particular, a paper by Abbott[16] summarizes 10 years of Parrondo's paradox. Furthermore, the idea that "lose" + "lose" = "win" has been extended to "chaos" + "chaos" = "periodic" in one-dimensional (1-D) maps [17–19]. For the logistic map, we have considered the case where the alternation of undesirable dynamical behaviors yields a desirable behavior in the context of seasonality, where we alternated the parameter values between even and odd iterations. In an extension of the so-called Parrondian games, we analyzed the dynamics of the 1-D logistic map, where we represented two seasons by alternating two relevant parameter values. For example, the alternation between a parameter that would drive the logistic map to extinction and a parameter that would drive the logistic map to chaos yielded stable oscillations. In the context of population dynamics, we have considered cases where "undesirable" + "undesirable" = "desirable" dynamical behaviors occur as a result of a simple alternation of parameters [20–23].

In general, the search for "undesirable" parameter values has thus far been limited to 1-D maps [17–23]. Here, for the first time, we analyze two-dimensional (2-D) maps and show that bifurcation diagrams allow us to find intervals of parameter values that, when used individually, yield chaotic dynamics and, when alternated, yield periodic orbits.

In our present discussion, we extend our modeling strategy to a couple of 2-D ecologically relevant maps and find that the "chaos" + "chaos" = "periodic" behaviors are not unique to 1-D maps [20–

* Corresponding author.
E-mail address: epeacock@williams.edu (E. Peacock-López).

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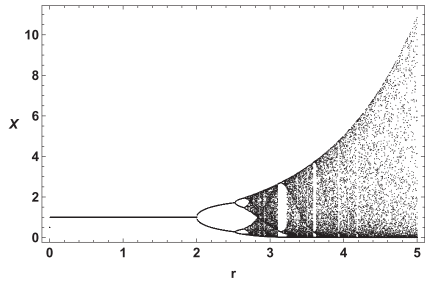

Fig. 1. Bifurcation diagrams for the Ricker model for two parameter ranges: $r \in [0, 5]$.

23]. In Section 2, we consider a modified Beddington, Free, and Lawton (BFL) map and, in Section 3, a system of two coupled Ricker maps. Finally, in Section 4, we discuss and summarize our results.

2. BFL model

In this section we analyze the BFL model,[24] that is a modified Nicholson–Bailey (NB) model[25] used to model the discrete dynamics of host (X)–parasitoid (Y) interactions. In particular, the model modifies the host's linear term in the NB model by a Ricker-type exponential dependency [26]. Therefore, we have a 2-D two-parameter map, given by the following equations:

$$x_{n+1} = X_n \exp[r(1 - X_n) - Y_n] = f_r(X_n, Y_n) \quad (1)$$

$$Y_{n+1} = cX_n(1 - \exp[-Y_n]) = g_c(X_n, Y_n) \quad (2)$$

where we refer to the host population as X and the parasitoid population as Y. For the BFL model, we note that if Y is decimated, the dynamics are described by the Ricker model, which is depicted in Fig. 1.

To simplify our analysis, we consider only the parasitoid growth parameter, c, as our bifurcation parameter, and without loss of generality we consider the analysis for $r = 2.0$. In other words, for each r values, we alternate the value of the growth rate of Y, as defined in the following equation:

$$X_{n+1} = \begin{cases} f_r(X_n, Y_n) & \text{if } n \text{ even} \\ f_r(X_n, Y_n) & \text{if } n \text{ odd} \end{cases} \quad (3)$$

$$Y_{n+1} = \begin{cases} g_{c_e}(X_n, Y_n) & \text{if } n \text{ even} \\ g_{c_o}(X_n, Y_n) & \text{if } n \text{ odd} \end{cases} \quad (4)$$

Note from Fig. 1 that the chosen r value yields P2 oscillation in the Ricker model, which can be observed if $Y = 0$ for a stable solution of the alternate map.

First, we depict in Fig. 2 the bifurcation diagram for the BFL model, Eqs. (1) and (2), for $r = 2.0$ and c the bifurcation parameter. We note that for small values ($c < 1$) and for large values ($c > 14$) of the bifurcation parameter the parasite population (Y) cannot survive and becomes extinct. Therefore, for Y extinction, we observe the expected Ricker P2 behavior. Moreover, in Fig. 3 we blow up three regions that we compare with the alternate map to identify values of undesirable parameter values that yield a periodic orbit when alternated. Next, we identify a value of c that yields an undesirable (non-periodic) orbit. For our analysis we consider, for the odd iterations, a value of 4, $c_O = 4.0$, which alternates with parameter values of the even iterates, c_E. Therefore, our bifurcation parameter is c_E, which we vary from 0 to 15, so we can compare the bifurcation diagram with Fig. 2.

In Fig. 4 we depict the bifurcation diagram for the alternate BFL model, and we note some significant changes in the interval $c \in [11, 10]$. Therefore, we can easily pick values that when alternated yield a periodic orbit. In this case, let us consider $c_E = 10.75$ and consider the three different orbits in Fig. 5. The first part (Fig. 5a) shows a chaotic behavior in the population of the host for $c = 10.75$, while for $c = 4$ the population shows a quasiperiodic orbit (Fig. 5b). However, when we alternate the values, we obtain a stable periodic behavior (Fig. 5c). In the periodic orbit, the populations show two spikes almost like a bursting oscillation. At these larger values the parasitoid, Y, should spike to a maximum value before it decimates the X population and at low X, the Y population also declines. Therefore, the host shows spikes, like a burst, the parasitoid shows single spikes, and, overall, the orbit shows a period ten (P10).

In general, it is straightforward to select parameters and construct the bifurcation diagrams for the alternate-Parrondian map. For example, let us choose $c_E = 10$, which is a high value for the parasitoid growth rate constant. In this case, we present Fig. 6, which has two windows of the full bifurcation diagram that can be compared with the diagrams in Fig. 3.

From the comparison, it is clear that $c_O = 4.6$ and $c_O = 8.16$ are parameters associated with quasiperiodicity and chaos, respectively. However, when the map alternates the values with $c_O = 10$, which is associated with chaotic orbits, the map yields periodic orbits. Therefore, we have two examples where "quasiperiodicity" + "chaos" = "periodic" and "chaos" + "chaos" = "periodic."

In Fig. 7, we show the chaotic and quasiperiodic orbits associated with $c = 10$ and $c = 4.60$, but when these values are alternated we obtain a periodic orbit P10. In this case, the orbit shows two-spike bursting, whereas in the chaotic case we observe three or two spikes. Thus, it seems that the quasiperiodic trajectory stabilizes the chaotic orbit, giving us the case of "quasiperiodicity" + "chaos" = "periodic."

In Fig. 8, we show two chaotic trajectories for $c = 10$ and $c = 8.16$, where both trajectories show bursting-type spikes. In the

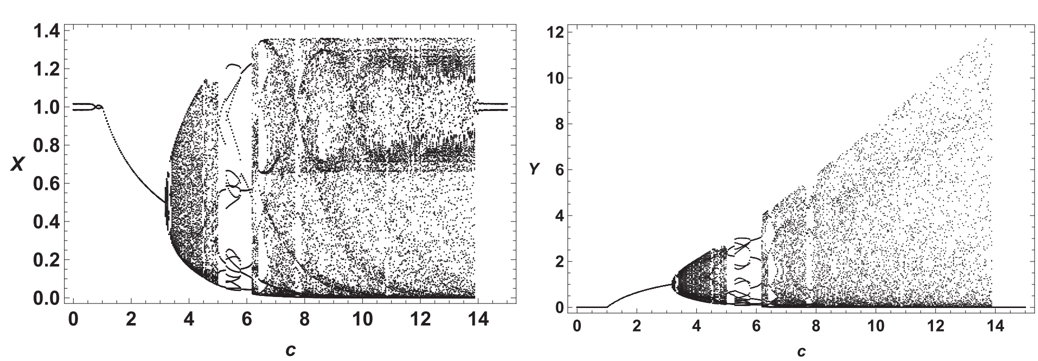

Fig. 2. Bifurcation diagram for the BFL model for: (a) the X population and (b) the Y population.