Chaos, Solitons and Fractals 107 (2018) 222–227

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Interaction of weak shocks in drift-flux model of compressible two-phase flows

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ARTICLE INFO

Article history:
Received 1 January 2017
Revised 7 November 2017
Accepted 28 December 2017

Keywords:
Drift-flux model
Wave interactions
Weak shock
Riemann problem

ABSTRACT

We consider the Riemann problem for an isothermal no-slip compressible gas-liquid drift-flux model of multi-phase flows governing two mass conservation equations and one mixture momentum equation. We establish a global result for the existence and uniqueness of a solution to the Riemann problem. We obtain one parameter family of elementary wave curves explicitly. Further, we derive a necessary and sufficient condition to ensure the existence of a shock wave or a simple wave for a 1-family and a 3-family of characteristics on the initial data in the solution of the Riemann problem. Finally, we prove the result of Von Neumann's concerning the overtaking of two weak shocks of same family.

1. Introduction

In the recent past, study of hyperbolic systems of partial differential equations (PDEs) have been the subject of great interest both from mathematical and physical point of view due to its applications in variety of fields such as magnetogasdynamics, astrophysics, engineering physics, multi-phase flows, aerodynamics and plasma physics etc. The Riemann problem was introduced by Riemann in [1] for system of hyperbolic conservation laws describing gas dynamics [2]. Riemann [1] initiated the concept of weak solutions and the method of phase plane analysis; Riemann's solutions reveal that the elementary waves of isentropic flows: shock waves and simple waves. His result was extended to adiabatic flows by Courant and Friedrichs [3], and a new kind of elementary wave, contact discontinuity (slip line), was added. In Lax's comprehensive discussion of such systems [4], it was proved that for strictly hyperbolic systems (*i.e.*, the eigenvalues of the Jacobian matrix are distinct), there is a unique solution of the Riemann problem provided the initial data given by two constant states U^l and U^r are sufficiently close (in a precise sense). Liu [5] proposed the entropy condition and solved the Riemann problem for general 2×2 conservation laws. Since then, a lot of interesting work has been contributed to the one dimensional Riemann problem for various hyperbolic systems of conservation laws (see, [6–10]); indeed, the Riemann problem has been playing an important role in all three areas of theory, applications and computation. Smith [11] investigated existence and uniqueness of solution of the Riemann problem, with arbitrary initial data U^l and U^r, for the equations of compressible fluid flow in one space variable.

The solution of the Riemann problem involves shock waves, centered simple waves and contact discontinuities which are called elementary waves. Another interesting feature of nonlinear systems is that interaction of elementary waves. Based on the solution of Riemann problem, the interaction of one dimensional elementary waves in gas dynamics has been studied by Courant and Friedrichs [3], Smoller [2], Chang and Hsiao [12]. Solution of Riemann problem and wave interactions for various physical problems have been discussed by Raja Sekhar et al. (see, [13–15]).

It has been demonstrated that considerable research has been devoted to the Riemann problem for two-phase flows with multiple components. Two-phase flows can be described by means of different models: mixture, drift (homogeneous or not), two-fluid or even multi-fluid models are currently used in industrial thermohydraulic codes. A general transient two phase problem is formulated by Goda et al. [16] using a two-fluid model or a drift-flux model and this depends on the degree of coupling between the phases. The basic concept of the drift-flux model is the consideration of two separate phases as a mixture phase. Therefore the fluid properties are represented by mixture properties making the drift-flux formulation simpler than the two-fluid formulation. The general expression of the drift-flux model, proposed by Zuber and Findlay [17], accounts for the effects of non-uniform flow and void distribution across the flow channel, and local relative velocity between phases. The drift-flux model on the other hand

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https://doi.org/10.1016/j.chaos.2017.12.030

is formulated by Ishii and Hibiki [18] considering the mixture as a whole rather than two phases separately. Numerical solution of one-dimensional drift-flux two-phase flow model have been studied in the literatures [19–21].

In this article, we consider the Riemann problem for an isothermal no-slip drift-flux model of multi-phase flows [17]. We establish the one-parameter families of curves for simple waves, shock waves and contact discontinuities. The main motivation of the present work is to study the existence and uniqueness of simple waves, shock waves and contact discontinuities. Lax's paper [4] leaves no doubt that such a solution exists, but it seems to us that there may be interest in a brief and explicit proof favorable to numerical computations. A necessary and sufficient condition is derived for arbitrary initial data for the existence of solution either in terms of shocks or simple waves or both. Further, we discuss the interaction of two weak shocks. This interaction has been treated by von Neumann [22].

The structure of this paper is as follows. In Section 2, we recall the drift-flux model. In Section 3, we derive elementary waves; simple waves, shock waves and contact discontinuities of the Riemann problem. In Section 4, we construct the one-parameter families of Riemann solution. In Section 5, we prove a result on global existence and uniqueness of solution of the Riemann problem and identify the possibilities for shocks or simple waves to occur in a 1-family and 3-family. Section 6 deals with the interaction of two weak shocks of same family and finally, in Section 7 we close the paper with the conclusions.

2. The drift-flux model

The basic concept of the drift-flux model is the consideration of two separate phases as a mixture phase. Therefore the fluid properties are represented by mixture properties making the drift-flux formulation simpler than the two-fluid formulation.

2.1. Governing equations

We consider a simple one dimensional model for two phases, with conservation of mass for each of the phases and a mixture momentum equation as follows [21];

$$\begin{aligned} &\frac{\partial}{\partial t}(\varrho_g\alpha_g) + \frac{\partial}{\partial x}(\varrho_g\alpha_g u_g) = m_g,\\ &\frac{\partial}{\partial t}(\varrho_l\alpha_l) + \frac{\partial}{\partial x}(\varrho_l\alpha_l u_l) = m_l,\\ &\frac{\partial}{\partial t}(\varrho_g\alpha_g u_g + \varrho_l\alpha_l u_l)\\ &+\frac{\partial}{\partial x}(\varrho_g\alpha_g u_g^2 + \varrho_l\alpha_l u_l^2 + \alpha_g p_g + \alpha_l p_l) = -q, \end{aligned} \tag{2.1}$$

where index g in the above system is referred to the gas and l to the liquid phase; x is the space coordinate and t is the time; ϱ_k, u_k, p_k, m_k, q and α_k are the density, velocity, pressure, mass source, wall friction momentum source and volume fraction for each phase, respectively. The volume fractions are subject to the constraint $\alpha_g + \alpha_l = 1$. This model is a so-called drift-flux model.

We also assume that there is no mass transfer between the phases, thus $m_g = m_l = 0$. The source term q is defined as $q = F_g + F_w$, where $F_g = g(\alpha_g\varrho_g + \alpha_l\varrho_l)\sin\theta$ represents the gravity, where g is the gravitational constant and θ is the inclination. The friction force term F_w takes into account the viscous forces and forces between the wall and fluids and is given by $F_w = \frac{32 v_{mix}\mu_{mix}}{d^2}$, where d is the inner diameter, $v_{mix} = \alpha_g u_g + \alpha_l u_l$ is the mixture average velocity, $\mu_{mix} = \alpha_g\mu_g + \alpha_l\mu_l$ and the viscosities for liquid and gas are assumed to be $\mu_l = 5\times 10^{-2}[Pas]$ and $\mu_g = 5\times 10^{-6}[Pas]$, respectively. To make (2.1) as a system of conservation laws we assume that $q = 0$.

2.1.1. Thermodynamic submodels

In this work, we assume that both gas and liquid phases are compressible. Dynamic mass and energy transfers are neglected. For each component the pressure for isothermal flows has the form

$$p = p_i(\varrho_i) = a_i\varrho_i, \quad i \in \{g, l\}, \tag{2.2}$$

where the positive constant a_i is the compressibility factor.

A *general pressure law* $p(\rho_1, \rho_2)$ can be derived as follows [23]: The relation $\alpha_g + \alpha_l = 1$ can then be written as $\frac{\rho_1}{\varrho_1} + \frac{\rho_2}{\varrho_2} = 1$, where $\rho_1 = \varrho_l\alpha_l$ and $\rho_2 = \varrho_g\alpha_g$. Hence, using (2.2), we have

$$p(\rho_1, \rho_2) = a_1\rho_1 + a_2\rho_2, \tag{2.3}$$

where $a_1 = a_l$ and $a_2 = a_g$.

2.1.2. Hydrodynamic submodels

The most important aspect of the model is the *hydrodynamic closure law*, which is commonly expressed in the following general form $u_g - u_l = \Phi(p, \alpha_g, u_g)$, which is also known as the slip relation.

To perform mathematical analysis for the Riemann problem of the current drift-flux model then, certain simplifying assumptions need to be made, leading to more simplified governing equations. More specifically, a common assumption within this context of drift-flux modeling is due to the no-slip condition, that is, $\Phi = 0$. For simplicity, therefore, the phase velocities are assumed to be in equilibrium $u = u_g = u_l$.

2.2. The Riemann problem

With the assumptions made in preceding Sections, the system (2.1) can be written in the following conservative form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \tag{2.4}$$

with

$$U = \begin{pmatrix} \rho_1 \\ \rho_2 \\ (\rho_1+\rho_2)u \end{pmatrix}, \quad \mathbb{F}(U) = \begin{pmatrix} \rho_1 u \\ \rho_2 u \\ (\rho_1+\rho_2)u^2 + p \end{pmatrix}$$

are the conservative variables and fluxes, respectively.

The Riemann problem for the current model, therefore, is formulated as a special initial value problem given by (2.4) for a gas-liquid two-phase flows with discontinuous initial data as

$$U(x,0) = \begin{cases} U^l, & \text{if } x < 0,\\ U^r, & \text{if } x > 0, \end{cases} \tag{2.5}$$

where U^l and U^r are constants.

To further establish the necessary mathematical framework for the current model equations, we write system (2.4) in the following alternative quasi-linear form of primitive variables $V = [\rho_1, \rho_2, u]^T$, where T denotes transposition, as

$$\frac{\partial V}{\partial t} + M(V)\frac{\partial V}{\partial x} = 0,$$

and the Jacobian matrix $M(V) = \frac{\partial F}{\partial V}$ is given by

$$\begin{pmatrix} u & 0 & \rho_1 \\ 0 & u & \rho_2 \\ \frac{a_1}{\rho_1+\rho_2} & \frac{a_2}{\rho_1+\rho_2} & u \end{pmatrix}.$$

Then one can observe that the Jacobian matrix has three real eigenvalues

$$\lambda_1 = u - w, \qquad \lambda_2 = u \quad \text{and} \quad \lambda_3 = u + w, \tag{2.6}$$