Chaos, Solitons and Fractals 106 (2018) 94–106

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Recursive sequences in the Ford sphere packing

Hui Li [a,*], Tianwei Li [b]

[a] Computer Science Department, Beijing University of Chemical Technology, Beijing 100029, China
[b] Tan Er Lane Primary School, No. 11, Tan Er Lane, Xicheng District, Beijing, 100051, China

ARTICLE INFO

Article history:
Received 29 July 2017
Revised 8 November 2017
Accepted 8 November 2017
Available online 20 November 2017

Keywords:
Apollonian sphere packing
Ford circle
Farey sequence
Generating function
Group action
Golden ratio

ABSTRACT

An Apollonian packing is one of the most beautiful circle packings based on an old theorem of Apollonius of Perga. Ford circles are important objects for studying the geometry of numbers and the hyperbolic geometry. In this paper we pursue a research on the Ford sphere packing, which is not only the three dimensional extension of Ford circle packing, but also a degenerated case of the Apollonian sphere packing. We focus on two interesting sequences in Ford sphere packings. One sequence converges slowly to an infinitesimal sphere touching the origin of the horizontal plane. The other sequence converges at fastest rate to an infinitesimal sphere in a particular position on the plane. All these sequences have their counterparts in Ford circle packings and keep similar features. For example, our finding shows that the x-coordinate of one Ford circle sequence converges to the golden ratio gracefully. We define a Ford sphere group to interpret the Ford sphere packing and its sequences finally.

1. Introduction

The problem of finding the circles tangent to three given circles was first studied and solved by Apollonius of Perga, after whom the Apollonian packing are also named. As a fractal, the Apollonian packing arise by successively filling the interstices between mutually tangent circles with further tangent circles. Descartes stated that for every four mutually tangent circles, the radii of the circles satisfy a certain quadratic equation. Frederick Soddy rediscovered the same equation in 1936 [1]. It is possible for every circle in the Apollonian packing to have integer radius of curvature [2], and this crucial packing was known as integral Apollonian circle packings [3].

Particularly, one of the generating circles may be replaced by a straight line of infinite radius. In this construction, the mutually tangent circles that are also tangent to the straight lines form a family of Ford circles introduced by Ford [4]. Ford circles are an important object of study in the geometry of numbers and hyperbolic geometry and has stimulated much further work. The Farey sequence F_Q is in bijection with the set of Ford circles tangent to the real line at points in the interval [0, 1].

Generalizations of Descartes configurations can be made from two-dimensional space to three-dimensional space and beyond. A Descartes configuration in R^n consists of $n+2$ mutually tangent $(n-1)$-spheres in n-dimensional Euclidean spaces. Proofs of the n-dimensional Soddy-Gosset theorem (the generalization of Descartes' theorem) first appeared in Pedoe's paper [5]. Particularly, a three-dimensional equivalent of the Apollonian packing is an Apollonian sphere packing.

Ford sphere packing is the degenerated case of the Apollonius sphere packing. It is constructed from three identical spheres on a horizontal plane of zero curvature. Ford sphere packing is also the three-dimensional equivalent of the Ford circle packing. Two mutually tangent circles become three mutually tangent sphere, and one horizontal line becomes one horizontal plane.

Fractal dimension is a measure in fractal geometry to describe how much space a fractal fills. Boyd [6] determined that the residual set dimension of Apollonian packing should lie between 1.300197 and 1.314534. Herrmann [7] calculated the fractal dimensions for two types of space-filling packing. The fractal dimension of the first type ($n=m=\infty$) agrees well with the bound in [6]. Mcmullen [8] presented an eigenvalue algorithm to compute the Hausdorff dimension of the Apollonian gasket. Athreya et al. [9] numerically computed the radial density for Apollonian circle packing, and extended the computation method to the Soddy sphere packing. Marszalek [10] calculated the total area of Ford circles in the range of $0 \le \frac{p}{q} \le 1$. Reis et al. [11] presented a systematic algorithm to estimate the maximum packing density of spheres when the grain sizes are drawn from an arbitrary size distribution. Borkovec et al. [12] yielded the estimate for the fractal dimension of Apollonius sphere packing, by adding up all spheres of radius greater than 2^{-19} contained in the Apollonian packing of the unit sphere.

* Corresponding author.
E-mail address: ray@mail.buct.edu.cn (H. Li).

https://doi.org/10.1016/j.chaos.2017.11.012
0960-0779/

Table 1
Some computational results for the Apollonian packing.

Item	Value(≈)	References
Dimension of the Apollonian circle packing	1.305688	Mcmullen et al. [8]
Radial density of the Apollonian circle packing	0.9549	Athreya et al. [9]
Total area of Ford circles for $0 \le \frac{p}{q} \le 1$	0.8723	Marszalek [10]
Radial density of the Ford sphere packing	0.853	Athreya et al. [9]
Dimension of the Apollonian sphere packing	2.473946	Borkovec et al. [12]

We collect some computational results, together with the references, shown in Table 1.

The set of Descartes quadruples encodes geometric information of the Apollonian packings [13]. And the action of a discrete group which is isomorphic to the Lorentz group $O(3, 1)$ permits one to walk around on a fixed Apollonian packing, moving from one Descartes quadruple to another quadruple in the same packing [14]. The discrete group, known as the Apollonian group, was introduced by Hirst [15] in 1967, and was later used in [16–18] for studying Apollonian packings. Graham et al. [19] introduced two more related finitely generated groups in $Aut(Q_D)$, the dual Apollonian group produced from the Apollonian group by a "duality" conjugation, and the super-Apollonian group which is the group generated by the Apollonian and dual Apollonian groups together. Besides, Graham et al. [20] introduced n-dimensional analogues of the Apollonian group, the dual Apollonian group and the super-Apollonian group.

Short [21] gave an elementary geometric proof using Ford circles that the convergents of the continued fraction expansion of a real number α coincide with the rationals that are best approximations of the second kind of α. Imamoglu et al.[22] showed that there exists exactly a unique zero of Eisenstein series inside each Ford circle. Percolation on a fractal like a circle or sphere packing is also an interesting topic [23].

One direct application of the circle packings is space-filling bearings, where the rotating cylinders of bearings corresponding to the circles in packings. Herrmann [7] showed that any even number of bearings could rotate on each other without slip and with the same tangential velocity. Oron and Herrmann [24] provided an algorithm to construct these bearings and explained the uniqueness of such construction. Baram and Herrmann [25] extended the design principles to the three-dimensional bearings. Stger and Arajo [26] performed an experiment showing how to impose any slip-free rotation state by only controlling two spheres. Arajo et al. [27] found that space-filling bearings can be perceived as complex network realizations of oscillators with asymmetrically weighted couplings. Furthermore, Satija [28] unveiled a mapping between an integral Apollonian circle packing and a quantum fractal that describes an iconic condensed matter problem of electrons moving in a two-dimensional lattice in a transverse magnetic field.

The primary goal of this paper is to conduct research on sequences in the Ford sphere packing. Sequences play an important role in linear algebra, analysis and topology. So far the only sequence in Ford circles has received attention is Farey sequence. If $0 \le p/q < 1$ then the Ford circles that are tangent to $C_{p/q}$ are precisely the Ford circles for fractions that are neighbours of p/q in some Farey sequence.

In this paper, we will focus on two other types of sequence. These sequences exist both in the Ford circle packings and the Ford sphere packings. The first sequence is a monotonic decreasing and slowly convergent sequence. The second sequence, by contrast, converge to a particular point at the fastest rate. To the best of our knowledge, the study on this topic has not emerged yet.

We do not give an extension of these sequences to four or higher dimensions, because in dimensions $n \ge 4$ the spheres in any such ensemble overlap and no longer correspond to a packing, as shown by Lemma 4.1 in [20].

The paper is organized as follows:

In Section 2, we provide a brief background, some mathematical notation and concepts about the Apollonian packing and Ford circles. In Section 3, we construct the Ford sphere packing and define the Ford sphere group in detail. We discuss the slowly converging sequence of the Ford circle packing in Section 4, and find a similar sequence in the Ford sphere packing thereafter. We show in Section 5 that there is another recursive sequence with fastest convergence rate in the Ford sphere packing. Naturally we explore its counterpart in the Ford circle packing. In Section 6, we describe the Ford sphere group and its group action to interpret transformations of Ford sphere configurations. Section 7 concludes the paper finally.

2. Preliminaries and main results

We start in terms of the circles' oriented curvatures (or signed curvature). The oriented curvature of a circle is defined as $b = \pm\frac{1}{r}$, where r is its radius. The smaller a circle, the larger is the magnitude of its curvature, and vice versa. The minus sign in $b = \pm\frac{1}{r}$ applies to an internally tangent circle that circumscribes the other circles.

René Descartes found a relation between the radii for four mutually disjoint tangent circles which is now called the Descartes circle theorem.

Theorem 1 (Descartes circle theorem *[29]*)**.** If four circles are tangent to each other at six distinct points, and the circles have oriented curvatures b_i (for $i = 1, \cdots, 4$)*, then the quadruple of mutually touching circles satisfy the Descartes equation:*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2\left(b_1^2 + b_2^2 + b_3^2 + b_4^2\right).$$

Descartes circle theorem also applies to a line (a degenerate circle with zero curvature) and three circles, or two lines and two circles that are all mutually tangent.

Any quadruple (b_1, b_2, b_3, b_4) satisfying this equation is defined as a Descartes quadruple. And a set of four mutually tangent circles is called a **Descartes configuration**.

Starting from any Descartes configuration, we can recursively construct an infinite circle packing, in which new circles are added which are tangent to three of the circles that have already been placed and have interiors disjoint from any of them. The infinite packing obtained in the limit of adding all possible such circles is called an **Apollonian packing** (Fig. 1).

The Descartes circle theorem gives a quadratic equation for the Descartes quadruple $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$, which can be rewritten as

$$\boldsymbol{b}^T \boldsymbol{Q}_D \boldsymbol{b} = 0 \tag{1}$$

where

$$\boldsymbol{Q}_D = \boldsymbol{I} - \frac{1}{2}\boldsymbol{e}\boldsymbol{e}^T = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \qquad \boldsymbol{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Eq. (1) is the Descartes quadratic form of the Descartes circle theorem.

Given an oriented circle C with center (x, y) and oriented curvature b, its **curvature-center coordinates** can be defined by the 1×3 row vector $m(C) := (b, bx, by)$. The curvature-center coordinates of a straight line can be defined as $m(H) := (0, \boldsymbol{h})$, where $\boldsymbol{h} = (h_1, h_2)$ is the unit normal vector of the line.

Theorem 2 (Extended Descartes theorem [29])**.** Given a Descartes configuration D of four oriented circles with oriented curvatures (b_1, b_2, b_3, b_4) *and centers* $\{(x_i, y_i): 1 \le i \le 4\}$,