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On distributional chaos in non-autonomous discrete systems

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ABSTRACT

This paper studies distributional chaos in non-autonomous discrete systems generated by given sequences of maps in metric spaces. In the case that the metric space is compact, it is shown that a system is Li–Yorke δ-chaotic if and only if it is distributionally δ'-chaotic in a sequence; and three criteria of distributional δ-chaos are established, which are caused by topologically weak mixing, asymptotic average shadowing property, and some expanding condition, respectively, where δ and δ' are positive constants. In a general case, a criterion of distributional chaos in a sequence induced by a Xiong chaotic set is established.

1. Introduction

In the present paper, we study the following non-autonomous discrete system (simply, NDS):

$$x_{n+1} = f_n(x_n), \quad n \geq 0, \tag{1.1}$$

where f_n: $X \to X$ is a map, and (X, d) is a metric space with metric d.

When $f_n = f$ for each $n \geq 0$, (1.1) is the following autonomous discrete system (simply, ADS):

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{1.2}$$

where f: $X \to X$ is a map.

Note that the ADS (1.2) is governed by the single map f while the NDS (1.1) is generated by iteration of a sequence of maps $\{f_n\}_{n=0}^{\infty}$ in a certain order. Thus, it is more difficult to study dynamical behaviors of NDSs than those of ADSs in general. However, many complex systems occurring in the real world problems such as physical, biological, and economical problems are necessarily described by NDSs. For example, the well-known logistic system

$$x_{n+1} = rx_n(1 - x_n), \quad n \geq 0,$$

describes the population growth under certain conditions. Note that the parameter r will vary with time when the natural environment changes. So it is more reasonable to describe the population growth in this case by the following non-autonomous system:

$$x_{n+1} = r_n x_n(1 - x_n), \quad n \geq 0.$$

Hence, it is meaningful to study NDSs and many mathematicians focused on complexity of NDSs in recent years [4,5,8,12,18,19,24,25,27,31–33,38,44].

Li and Yorke firstly introduced the concept of chaos in their famous work "period three implies chaos" [13], which has activated sustained interest in the frontier research on discrete chaos theory. Later, various definitions of chaos were developed, such as Devaney chaos [7], Auslander–Yorke chaos [2], generic chaos [29], dense chaos [30], Xiong-chaos [40], distributional chaos [23], and so on. These definitions all focus on the complex trajectory behavior of points. The concept of distributional chaos (it was called strong chaos then) was introduced by Schweizer and Smital from the perspective of probability theory for a continuous map in a compact interval [23]. Since then, it has evolved into three mutually nonequivalent versions of distributional chaos for a map in a metric space: DC1, DC2, and DC3 [3,28]. DC1 was the original version of distributional chaos introduced in [23], and it is the strongest one among these three definitions, while DC2 and DC3 are its generalizations. In our present paper, we pay attention to DC1 for NDSs. So please remember that distributional chaos always

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means DC1 in the rest of the present paper without special illustration. We shall study properties of DC2 and DC3 for NDSs in our forthcoming papers.

For ADSs, many elegant results about distributional chaos have been obtained [1,14,16,17,20,21,34–37,42]. Most of them are about relations between distributional chaos and other concepts of chaos. For example, Oprocha showed that neither topologically weak mixing nor Devaney chaos implies distributional chaos using one-sided symbolic dynamical systems [20]. Then, Li et al. proved that Devaney chaos implies distributional chaos for continuous maps with shadowing property in compact metric spaces [16]. Shadowing property (also called pseudo-orbit tracing property), is a property that a pseudo-orbit can be traced by a true trajectory, which has been extensively studied (see, for example, [6,15,39]). The asymptotic average shadowing property was introduced in [10]. Recently, Wang et al. proved that the asymptotic average shadowing property implies distributional chaos for a continuous map with two almost period points in a compact metric space [37]. Note that Xiong-chaos, introduced in [40], is induced by a topologically mixing map. In 2002, Yang introduced the notion of distributional chaos in a sequence, and investigated some relationships of topologically weak mixing, Xiong-chaos and distributional chaos in a sequence for a continuous map in a locally compact metric space [42]. In 2007, Wang et al. established a criterion of distributional chaos under some expanding conditions, and also proved that Li–Yorke chaos is equivalent to distributional chaos in a sequence for a continuous map in a compact interval [35]. Later, Li and Tan generalized their result and proved that Li–Yorke δ-chaos is equivalent to distributional δ-chaos in a sequence for a continuous map in a compact metric space [14].

For NDSs, Kolyada and Snoha extended the concept of topological entropy for ADSs to NDSs and studied its properties [14]. Later, many scholars investigated chaos of NDSs [4,5,8,18,19,24,27,31–33,38,44]. For example, Tian and Chen extended the concept of Devaney chaos to NDSs in [33]. Then, Chen with the second author in the present paper generalized related concepts of chaos, such as topological transitivity, sensitivity, chaos in the sense of Li–Yorke, Wiggins, and Devaney to general NDSs, and established a criterion of Li–Yorke chaos induced by strict coupled-expansion for a certain irreducible transitive matrix [27]. In 2012, Balibrea and Oprocha investigated relations of Li–Yorke chaos with positive topological entropy and topologically weak mixing for NDSs [4]. Recently, we studied whether transitivity and density of periodic points imply sensitivity in the definition of Devaney chaos [44]. However, there are only a few results about distributional chaos for NDSs [8,18,38]. To the best of our knowledge, there are no criteria of distributional chaos established for NDSs in the current literature. Motivated by the above works, we shall study distributional chaos in NDSs and try to establish its criteria in the present paper.

The rest of the paper is organized as follows. Section 2 presents some related concepts and useful lemmas. It is divided into four parts for convenience. In Section 2.1, some basic concepts are given. In Sections 2.2 and 2.3, several lemmas about density of a sequence and some properties of one-sided symbolic dynamical systems are recalled, respectively. They are useful to Sections 3 and 4. Several kinds of relations for NDSs are introduced in Section 2.4. Section 3 reveals relations between Li–Yorke δ-chaos and distributional δ-chaos in a sequence; and Section 4 gives out three criteria of distributional δ-chaos in compact metric spaces, which are caused by topologically weak mixing, asymptotic average shadowing property, and some expanding condition, respectively. Finally, a criterion of distributional chaos in a sequence induced by a Xiong chaotic set is established in general metric spaces in Section 5.

2. Preliminaries

In this section, some related concepts and useful lemmas are presented.

2.1. Some basic concepts

For any fixed $x_0 \in X$, $\{x_n\}_{n=0}^{\infty}$ is called the (positive) orbit of system (1.1) starting from x_0, where $x_n = f_0^n(x_0)$ and $f_0^n := f_{n-1} \circ \cdots \circ f_0$ for $n \geq 1$. For convenience, by f_0^0 denote the identity map on X, $f_0^{-n} := (f_0^n)^{-1}$, and $f_{0,\infty} := \{f_n\}_{n=0}^{\infty}$. By $\overline{A}$ denote the closure of a subset $A \subset X$, and by $\mathbf{N}$ and $\mathbf{Z}^+$ denote the set of all nonnegative integers and that of all positive integers, respectively.

Let $P = \{p_n\}_{n=1}^{\infty} \subset \mathbf{N}$ be an increasing sequence, x, $y \in X$, and $\epsilon > 0$. Denote

$$F_{x,y}^{*}(\epsilon, P) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\epsilon)}\left(d(f_0^{p_i}(x), f_0^{p_i}(y))\right),$$

$$F_{x,y}(\epsilon, P) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\epsilon)}\left(d(f_0^{p_i}(x), f_0^{p_i}(y))\right),$$

where $\chi_{[0,\,\epsilon)}$ is the characteristic function defined on the set $[0, \epsilon)$. $F_{x,y}^{*}$ and $F_{x,\,y}$ are called the upper and lower probability distributional functions, respectively.

Definition 2.1. Let $D \subset X$ contain at least two distinct points. Then D is called a distributionally scrambled set in a sequence P of system (1.1) if, for any x, $y \in D$ with $x \neq y$,

(i) $F_{x,y}^{*}(\epsilon, P) = 1$ for any $\epsilon > 0$,
(ii) $F_{x,y}(\delta_{x,y}, P) = 0$ for some $\delta_{x,\,y} > 0$,

and then the pair (x, y) is called a distributionally $\delta_{x,\,y}$-scrambled pair in the sequence P. Further, D is called a distributionally δ-scrambled set in the sequence P if there exists $\delta > 0$ such that (i) holds and $F_{x,y}(\delta, P) = 0$ for any x, $y \in D$ with $x \neq y$. Moreover, if $P = \mathbf{N}$, then D is called a distributionally scrambled set (distributionally δ-scrambled set), and $F_{x,y}^{*}(\epsilon, \mathbf{N})$ and $F_{x,\,y}(\delta, \mathbf{N})$ are briefly denoted by $F_{x,y}^{*}(\epsilon)$ and $F_{x,\,y}(\delta)$, respectively.

Definition 2.2. If system (1.1) has an uncountable distributionally scrambled set (distributionally δ-scrambled set) in a sequence P, then it is said to be distributionally chaotic (distributionally δ-chaotic) in the sequence P. Further, if $P = \mathbf{N}$, then system (1.1) is said to be distributionally chaotic (distributionally δ-chaotic).

Definition 2.3 ([27], Definition 2.7)**.** Let $S \subset X$ contain at least two distinct points. Then, S is called a Li–Yorke scrambled set of system (1.1) if, for any two distinct points x, $y \in S$, their corresponding orbits satisfy

(i) $\liminf_{n\to\infty} d(f_0^n(x), f_0^n(y)) = 0$,
(ii) $\limsup_{n\to\infty} d(f_0^n(x), f_0^n(y)) > 0$.

Further, S is called a Li–Yorke δ-scrambled set for some positive constant δ if, for any two distinct points x, $y \in S$, (i) holds and, instead of (ii), the following holds:

(iii) $\limsup_{n\to\infty} d(f_0^n(x), f_0^n(y)) > \delta$.

Definition 2.4. System (1.1) is said to be Li–Yorke chaotic (δ-chaotic) if it has an uncountable Li–Yorke scrambled set (δ-scrambled set). Further, system (1.1) is said to be densely Li–Yorke chaotic (δ-chaotic) if it has a densely uncountable Li–Yorke scrambled set (δ-scrambled set).

Remark 2.5. Definitions 2.1 and 2.2 generalize the concept of distributional chaos in a sequence for ADSs introduced in [35,42] to