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Periodicity, chaos and multiple coexisting attractors in a generalized Moore–Spiegel system

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ABSTRACT

Introduced in 1966 by Moore and Spiegel, the so called Moore–Spiegel system displays aperiodic dynamics that describes the irregular variability of the luminosity of the stars. The Moore–Spiegel system is defined by a jerk system with a single cubic nonlinearity that is responsible for the chaotic dynamics of the whole system. In this contribution, the dynamics of the generalized Moore-Spiegel system recently investigated by [Letellier and Malasoma, Chaos, Solitons & Fractals 69 (2014) 40–49] is considered. Some fundamental dynamical properties of the model such as fixed points, phase portraits, basins of attraction, bifurcation diagrams, and Lyapunov exponents are investigated. Analysis shows that chaos is achieved via period-doubling and symmetry restoring crisis scenarios. One of the major results of this work is the finding of various windows in the parameters' space in which two, three, four or six different attractors coexist, depending solely on the choice of initial conditions. This unusual and striking phenomenon has not yet been reported previously in the Moore–Spiegel system, and thus deserves dissemination. Compared to some lower dimensional systems capable of six disconnected coexisting periodic and chaotic attractors reported to date, the Moore–Spiegel system represents one of the simplest and the most 'elegant' paradigms. Some PSIM based simulations of the nonlinear dynamics of the system are carried out to validate the results of theoretical analyzes.

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1. Introduction

Multistability is one of the most striking phenomena in the dynamics of nonlinear and chaotic systems. Considered as the coexistence of two or more attractors at a point of operation in phase space, multistability is characterized by its sensitivity to initial conditions [1]. It is possible to leave from one attractor to another by just changing the initial conditions. Correspondingly, the complexity of the system increases with the number of solutions that coexist. The phenomenon of multistability sometimes appears as an undesired behavior in the application of chaotic oscillators due to the change of unconditional state of the attractors [2–6]. The study for this type of phenomenon in nonlinear dynamic systems thus becomes a major issue in the field of science. Recently, some researchers have studied this question in order to analyze multistability in systems described by simple mathematical models [7–9].

In this contribution, we consider the dynamics of the well-known Moore and Spiegel system paying particular attention on the mechanism of chaos generation as well as the occurrence of multiple coexisting attractors. Before focusing our attention on the generalized Moore and Spiegel oscillator under consideration, let us briefly recall the interesting works related to the original Moore and Spiegel oscillator. The original form of mathematical model of this oscillator was introduced by Moore and Spiegel [10] in the Jerk form as $\dddot{x} + \ddot{x} + (T - R + Rx^2)\dot{x} + Tx = 0$, where R and T are parameters related to Prandtl, Taylor and Rayleigh numbers. The model represents a small fluid element oscillating vertically in a temperature gradient with a linear restoring force. This element exchanges heat with surrounding fluid and its buoyancy depends upon temperature. In the latter literature, the authors demonstrated that this oscillator can generate the aperiodic oscillations describing the irregular variability of the luminosity of the stars. Nineteen years later, Auvergne and Baglin [11] proposed a model of the motion of the ionization zone of a star by an equation derived from this Moore and Spiegel oscillator. In 1997, Balmforth and his colleague studied the bifurcations and synchronization (in the sense of Pecora and Carroll) in the Moore–Spiegel oscillator [12]. From this

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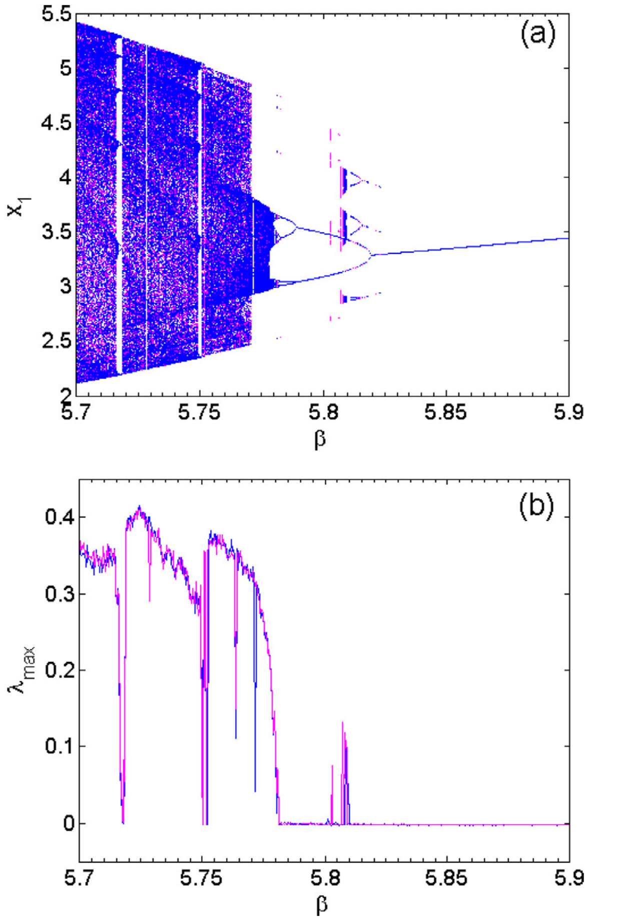

Fig. 1. Bifurcation diagram (*a*) showing local maxima of the coordinate $x_1$versus β and the corresponding graph (*b*) of largest Lyapunov exponent (λ_{max}) plotted in the range $5.7 \le \beta \le 5.9$. A positive exponent ($\lambda_{max} > 0$) indicates chaos while regular states are characterized with negative values of Lyapunov exponent ($\lambda_{max} < 0$).

synchronization study, the authors concluded that the Moore and Spiegel system could not be synchronized using two coordinates, but may be possible with the variable *x*. However, they showed that synchronization of this coordinate fails on certain parameter ranges. The determination of where synchronization is successful boils down to a numerical exercise. More recently, Letellier and Malasoma studied the effect of parity of nonlinearity on the generalized model of Moore and Spiegel [13]. The system is described by a relatively simple "Jerk" equation as $\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^n\dot{x} = 0$. It follows that in these generalized Jerk equations, the single nonlinear term has a parity which depends on $x^n\dot{x}$. The system has an inversion symmetry when n is even and no symmetry property when n is odd. They show also that the topology of chaotic solutions only depends on the parity of n and the value of n only affects the possibility to develop the chaotic solution. Motivated by the above-mentioned results, and provided the importance of the Moore–Spiegel system, this paper focuses on the dynamics of generalized Moore and Spiegel system with particular emphasis on the occurrence of multiple attractors. Some windows in the parameter space are found in which two, four or six distinct non static coexisting attractors are reported. We would like to stress that, to the best of the author's knowledge, a situation involving the coexistence of multiple (up to six or four) non-static (i.e. oscillatory) attractors in Moore–Spiegel system is not reported so far in the relevant literature.

The rest of the paper is arranged as follows. Section 2 describes the mathematical model of the system under investigation and highlights possible symmetries and implications. The stability of the fixed point is analyzed. In Section 3, the bifurcation structures of the system are investigated numerically yielding some windows (in the parameter space) of multiple coexisting attractors. In Section 4, an appropriate electronic circuit describing the Moore–Spiegel system is designed. Some PSIM simulations of different kinds of coexisting attractors are carried out to confirm the numerical analysis. Finally in Section 5, we summarize our results and draw the conclusions of this work.

2. Description and analysis of the model

In this section, the model of the generalized Moore and Spiegel system is introduced and described. The existence of attractive sets (i.e. attractors) is examined by computing the volume contraction rate of the model. The symmetry properties and possible implications are discussed. The stability and nature of the fixed point are studied based on the Routh stability criterion.

2.1. The model

The mathematical model of the generalized Moore and Spiegel system [14] considered in this work is expressed by the following simplified form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \alpha x_3 \\ \dot{x}_3 = -x_3 + \varepsilon x_2 - x_2 {x_1}^2 - \beta x_1 \end{cases} \tag{1}$$

where the dot denotes differentiation with respect to time, $\alpha, \beta, \varepsilon \ge 0$ are tuneable parameters. It is obvious that system (1) has a single nonlinearity in which two state variables (namely x_1 and x_2) are involved. The presence of this nonlinearity accounts for the chaotic behavior of the whole system. Equivalently system (1) can be written as a jerk equation:

$$\dddot{x} + \ddot{x} + \left(-\varepsilon + x^2\right)\alpha\dot{x} + \alpha\beta x = 0. \tag{2}$$

This latter expression is typical of 'elegant' jerk dynamic systems. Mention that system (1) represents one of the simplest autonomous 3D system reported to date, capable of displaying six disconnected chaotic and periodic attractors (see Section 3 and 4) depending solely on the initial conditions [15–17]. Finally, one can note that system (4) is dissipative with an exponential contraction rate:

$$\frac{dV}{dt} = \exp\left(-c\tau\right) \tag{3}$$

provided that

$$\nabla V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -c < 0 \tag{4}$$

at any given point (x_1, x_2, x_3) of the state space. Consequently, the general condition of dissipativity related to the existence of attractive sets in our model is satisfied [19–21].

2.2. Symmetry

System (1) is unchanged under the coordinate transformation: $(x_1, x_2, x_3) \Leftrightarrow (-x_1, -x_2, -x_3)$. This means that if (x_1, x_2, x_3) is a solution of system (1) for a given set of parameters values, then $(-x_1, -x_2, -x_3)$ is also a solution for the same parameters set. A symmetric solution is a solution of (1) that is invariant under the above transformation [14,18]; otherwise it is an asymmetric solution. The single equilibrium point $E_0(0, 0, 0)$ is a trivial symmetric static solution. As a result, attractors in phase space must be symmetric by inversion with respect to the origin; otherwise they must appear in pair, to satisfy the exact symmetry of the model equations. This exact symmetry is interesting as it can be used to