Bifurcation and stability analysis of a ratio-dependent predator-prey model with predator harvesting rate

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ABSTRACT

In this paper, we study the bifurcation and stability of a ratio-dependent predator-prey model with non-constant predator harvesting rate. The analysis is carried out both analytically and numerically. We determine stability and dynamical behaviours of the equilibria of this system and characterize codimension 1 and codimension 2 bifurcations of the system analytically. Our bifurcation analysis indicates that the system exhibits numerous types of bifurcation phenomena, including Fold, Hopf, Cusp, and Bogdanov–Takens bifurcations. We use the numerical software MATCONT, to compute curves of equilibria and to compute several bifurcation curves. We especially approximate a family of limit cycles emanating from a Hopf point. Our results generalize and improve some known results and show that the model has more rich dynamics than the ratio-dependent predator-prey model without harvesting rate.

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1. Introduction

In recent years, great attention has been paid to the management of renewable resources, because mankind is facing the problems about shortage of resources and a worsening environment. From the view of human needs, the exploitation of biological resources and harvest of population are commonly practised in the fields of fishery, wildlife, and forestry management. From those earlier results, we can see that harvesting has a strong impact on population dynamics, which may range from the rapid depletion to the complete preservation of biological population. Hence, harvesting has been studied widely (see [27]). The predator-prey system plays an important and fundamental role among the relationships between the biological populations. Many scholars have carried out the study of the prey-predator system with various functional responses, such as the Monod-type (see [8,12,13]), Holling-type (see [14–20]), Ivlev-type (see [21,23]), and so on. Predator-prey models with different functional responses or (and) harvesting are refined so as to better reflect the specific characteristics of the different populations or economical needs (see [10,23–25]). The proposed models usually depend on some parameters and are stud-

ied by bifurcation methods. Local stability of population models of two species governed by planar systems of first order autonomous differential equations can be studied by the well-known qualitative theory on phase portraits of planar systems [2,4,8,13–15,18,20]. The main aim of this paper is to study the pattern of bifurcation that takes place as we vary some of the model parameters. We especially focus on the biological implications of the found bifurcations. Most importantly, we show that the Hopf bifurcation plays, for various reasons, a crucial role. Ecological systems are complex because of the diversity of biological species as well as the complex nature of their interactions. We further analytically characterize the Bogdanov–Takens bifurcation which is an organising centre of the dynamics of the system. In this paper we rely heavily on advanced continuation and bifurcation techniques implemented in the software package MATCONT [1,5,6] to obtain results that cannot be obtained analytically.

MATCONT is a dynamical toolbox based on numerical continuation technique which is a well-understood subject [1,3,7]. This software computes a solution curve of equation $F(x) = 0$ for the system of the form

$$\frac{dx}{dt} = f(x, \alpha)$$

with $x \in \mathbb{R}^n$, $f(x, \alpha) \in \mathbb{R}^n$, and α a vector of parameters where equilibria, limit points, limit cycles, etcetera can be computed.

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MATCONT is compatible with the standard MATLAB ODE of differential equations. General descriptions, functionalities and the dynamical algorithms implemented in MATCONT can be found in [1,5,6].

We consider the ratio-dependent predator-prey model with nonconstant predator harvesting rate:

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{k}\right) - \dfrac{c_1 uv}{u + mv}, \\ \dfrac{dv}{dt} = v\left(-d + \dfrac{c_2 u}{u + mv}\right) - hv, \end{cases} \tag{1}$$

where $u(t)$ and $v(t)$ denote population densities of prey and predators at time t, respectively. The parameters r, K, c_1, m, d and c_2 are positive constants and present the prey intrinsic growth rate, the carrying capacity K in the absence of predation, capturing rate, half capturing saturation constant, the death rate of the predators and the conversion rate, respectively. The parameter $h \geq 0$ denotes the predator harvesting rate. The term $\frac{c_1 u}{u+mv}$ is called the ratio-dependent functional response of Holling type II. We note that the system (1) with $h = 0$ becomes the classic ratio-dependent predator-prey models without any harvesting rates studied in [8,11].

Using the transformation: $x = \frac{u}{K}$, $y = \frac{mv}{K}$ and $\bar{t} = rt$, one can write the system (1) into the following equivalent system

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x) - \dfrac{\alpha xy}{x + y}, \\ \dfrac{dy}{dt} = y\left(-\gamma + \dfrac{\beta x}{x + y}\right) - \delta v, \end{cases} \tag{2}$$

where $\alpha = \frac{c_1}{rm}$, $\beta = \frac{c_2}{r}$, $\gamma = \frac{d}{r}$ and $\delta = \frac{h}{r}$. The parameters α, β, γ and δ have the same biological meanings as c_1, c_2, d and h, respectively. The stability of the systems (1) and (2) was studied in [4,9,15]. It was shown in [15] that under suitable conditions on the parameters there are two positive equilibria of (2), namely, a predator free equilibrium and an interior equilibrium. Stronger sufficient conditions were given in [4] for the existence of positive equilibria of (1). It was shown without proofs in [9] that the predator free equilibrium of (2) is a saddle point or a stable equilibrium under $\beta > \gamma + \delta$ or $\beta < \gamma + \delta$, and the interior equilibrium can be a stable or unstable focus or node or center by comparing the parameter α and another number depending on β, δ and γ. The asymptotic stability of the equilibria of (1) was studied in [4], where some sufficient but complicated conditions on the parameters are given. The methods used in [4,9] are to determine the signs of the eigenvalues of the Jacobian matrices at the positive equilibria. The globally asymptotic stability of the classic ratio-dependent population models (1) with $h = 0$ were studied for example in [8,11], where a few sufficient conditions on the parameters were provided to ensure that (1) is globally asymptotic stability at the equilibria. We refer to [26,27] for the study on the stability in (2) with constant predator and prey harvesting rate and to [13,18,28] for the study on other population models with prey harvesting rate.

This paper is organized as follows: In Section 2, we consider the mathematical model and discussed some basic dynamical results like positivity, boundedness of solution and existence of possible equilibria. In Section 3, the stability of equilibria, Hopf bifurcation of the interior equilibrium point of the system is discussed. In this Section, we show that in a small neighbourhood of the interior equilibrium point, the system undergoes the Cusp bifurcation and Bogdanov–Takens bifurcation of codimension 2, when the parameters vary in a small neighbourhood of some parameter values. Numerical bifurcation results are included to support our analytical results in Section 4. The paper concludes with a brief discussion in Section 5.

2. Preliminaries

In this section, we shall present some preliminary results. The parameters α, β, γ and δ represent prey capturing rate, prey conversion rate, death rate of the predator and harvesting rate on predator, respectively. In particular, if $\delta = 0$, then (2) becomes the following classic ratio-dependent predator-prey model without any harvesting rate:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x) - \dfrac{\alpha xy}{x + y} = f(x, y), \\ \dfrac{dy}{dt} = y\left(-\gamma + \dfrac{\beta x}{x + y}\right) = g(x, y), \end{cases} \tag{3}$$

The system (2) is equivalent to (3) which is studied in [11]. We recall that $(x, y) \in R^2$ is an equilibrium of (2) if it satisfies $f(x, y) = 0$ and $g(x, y) = 0$. An equilibrium point (x, y) is said to be positive if x, $y \geq 0$ and be a positive interior equilibrium if x, $y > 0$.

Assume that $\alpha > 0$, $0 < \gamma < \beta$ and $\delta_1 < \delta < \beta - \gamma$, where $\delta_1 := \delta_1(\alpha, \beta, \gamma) = \max\{0, \beta - \gamma - \frac{\beta}{\alpha}\}$. Then the system (2) has a unique positive interior equilibrium (x_*, y_*), where $x_* = 1 - \alpha + \frac{\alpha(\gamma+\delta)}{\beta}$, $y_* = \frac{\beta-\delta-\gamma}{\gamma+\delta}$. For α, β, $\gamma > 0$, (3) has a positive equilibrium $E_0 = (1, 0)$. Assume that $0 < \gamma < \beta$ and $0 < \alpha \leq \frac{\beta}{\beta-\gamma}$, then (3) has a positive interior equilibrium (x_*, y_*), where $x_* = 1 - \alpha + \frac{\alpha\gamma}{\beta}$, $y_* = \frac{\beta-\gamma}{\gamma}$.

The Jacobian matrix of system (3) at $E_0 = (1, 0)$ takes the form:

$$\begin{aligned} J|_{E_0} &= \begin{pmatrix} 1 - 2x - \dfrac{\alpha y^2}{(x+y)^2} & -\dfrac{\alpha x^2}{(x+y)^2} \\ \dfrac{\beta y^2}{(x+y)^2} & -\gamma + 2\dfrac{\beta x^2}{(x+y)^2} \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\alpha \\ 0 & \beta - \gamma \end{pmatrix}. \end{aligned}$$

Since $det(J|_{E_0}) > 0$ and $tr(J|_{E_0}) = -1 < 0$, the equilibrium $E_0 = (1, 0)$ becomes a stable-node.

Lemma 1. *Let E_0 be an equilibrium of (3). Then the following assertions hold.*

(i) *If $det(J|_{E_0}) < 0$, then E_0 is a saddle of (3).*
(ii) *If $det(J|_{E_0}) > 0$, $tr(J|_{E_0}) \neq 0$ and $(tr(J|_{E_0}))^2 - 4det(J|_{E_0}) \geq 0$, then E_0 is a node of (3); it is stable if $tr(J|_{E_0}) < 0$ and unstable if $tr(J|_{E_0}) > 0$.*
(iii) *If $det(J|_{E_0}) > 0$, $tr(J|_{E_0}) \neq 0$ and $(tr(J|_{E_0}))^2 - 4det(J|_{E_0}) < 0$, then E_0 is a focus of (3); it is stable if $tr(J|_{E_0}) < 0$ and unstable if $tr(J|_{E_0}) > 0$.*
(iv) *$det(J|_{E_0}) > 0$, $tr(J|_{E_0}) = 0$, then E_0 is a center, or a focus of (3).*

It is evident that if $det(J|_{E_0}) > 0$, $tr(J|_{E_0}) < 0$, then E_0 is locally asymptotically stable.

3. Bifurcation analysis

In this section, we study the stability and bifurcation of positive equilibria. We first discuss the existence of Hopf and Bogdanov–Takens bifurcations of the system (3).

3.1. The Bogdanov–Takens (or double zero) bifurcation analysis

We first prove the unique positive equilibrium of system (3) is a Cusp of codimension 2, then discuss the Bogdanov–Takens bifurcation of this system. If $f(x) = 0$ and f $f'(x) = 0$ simultaneously