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A regularity statistic for images

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ABSTRACT

Measures of statistical regularity or complexity for time series are pervasive in many fields of research and applications, but relatively little effort has been made for image data. This paper presents a method for quantifying the statistical regularity in images. The proposed method formulates the entropy rate of an image in the framework of a stationary Markov chain, which is constructed from a weighted graph derived from the Kullback–Leibler divergence of the image. The model is theoretically equal to the well-known approximate entropy (ApEn) used as a regularity statistic for the complexity analysis of one-dimensional data. The mathematical formulation of the regularity statistic for images is free from estimating critical parameters that are required for ApEn.

1. Introduction

The study of complexity has been mainly concerned with time series generated by dynamical systems. Most well-known methods for discovering the nonlinear phenomena of time series are based on the concepts of chaos and nonlinear dynamics, and their applications can be found extensively across many disciplines of sciences, medicine, health, and engineering [1–5]. Yet relatively little effort has been made toward the development of methods for quantifying the complexity of image content, which inherently exists in many forms of this kind of data.

Representative and chronical reports on various measures of complexity of images include fractal surface measurement methods (isarithm, variogram, triangular prism) for characterizing the complexity of remote-sensing landscape images [6], measuring image complexity with information channel maximizing the mutual information [7], mean information gain for quantifying habitat change in a forest ecosystem [8], image complexity for steganalysis using the shape parameters of the generalized Gaussian distribution of wavelets [9], modeling of visual complexity based on fuzzy entropic distance functions [10], image complexity using independent component analysis (ICA) for content-based image retrieval [11], quantification of image complexity using singular value decomposition (SVD) [12], geostatistics and nonlinear dynamics analysis of biomedical signals [13], visual complexity using multiple parameters related to the mechanisms of visual processing [14], measure of image complexity based on compression quality [15], complexity measures for noise filtering [16], chaos analysis and nonlinear dynamics [17–19], fractal analysis for texture classification [20], and qualitative evaluation of visual complexity based on the psychology of artworks [21].

Given wide range of applications in different fields, the definition of complexity of an image is still not well-defined [14]. It is rather problem dependent as different methods introduce different standpoints for quantifying the complexity of images. The Shannon entropy of an image intensity histogram has been considered as a measure of image complexity, but this formulation does not consider the spatial distribution of pixels in images [7]. Other methods address the issue of image complexity in the context of image compression. The association between complexity and compression stems from the concept of the Kolmogorov complexity introduced in algorithmic information theory. The Kolmogorov complexity is known as the descriptive complexity of an object being equivalent to the length of the shortest computer program that produces the object as the output from basic elements [22]. However, in addition to the difficulty of the computation of the Kolmogorov complexity, it has been known that the Kolmogorov complexity is not associated with the underlying nature of visual appearance in images [11,14].

This paper presents a method for measuring image complexity in the context of a regularity statistic, which is a notion of complexity addressed by ApEn to quantify the pattern of regularity in one-dimensional data [23–26]. By using the image histogram, which conveys statistical information about the image intensity, each row or column of an image is then normalized to represent a finite probability distribution. The weights between image rows

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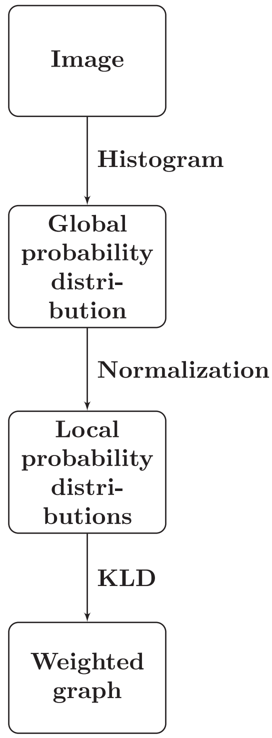

Fig. 1. Procedure for constructing a weighted graph of an image using its pixel value histogram information and KLD-based edge weights.

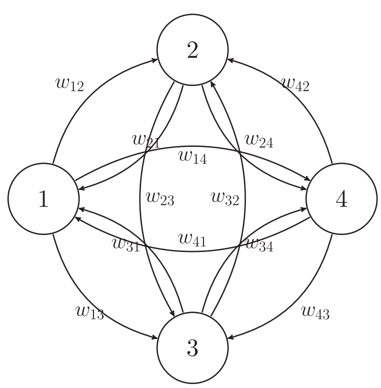

Fig. 2. A 4-state Markov chain of a KLD-weighted graph of an image, where states are either rows or columns of the image.

or columns can then be computed using the Kullback–Leibler divergence (KLD) [22] to construct a weighted graph of the image. By imposing constraints on the graph of the image as a stationary Markov chain, the entropy rate of the Markov chain, which is the weighted transition entropy, can be computed. Such an entropy rate has been proved to be equal to the regularity statistic obtained from ApEn [23]. The idea of mapping an image to a graph, whose vertices are pixels and edge weights are pairwise similarities, was proposed to construct a graph consisting of a subset of edges [27]. The formulation here derives a weighted graph whose vertices are either rows or columns of an image and edge weights are measures of differences between two probability distributions of the corresponding pairs of image rows or columns, and both separate sources of image information are combined as the entropy sum to represent the image regularity statistic.

The motivation for using the KLD to determine the weights of an image-based graph edges is based on several theoretical aspects. In comparison with other metric functions such as the Euclidean distance, the KLD conveys statistical meaning and is geometrically important, because its asymmetric property exists for a manifold of probability distributions while no distance functions can take on this measure [28]. Moreover, the KLD has three special properties that make them important in information processing [29]: (1) an information-theoretic function that satisfies the data processing inequality, (2) being an exponential rate of optimal classifier performance probabilities, and (3) its Hessian matrix is proportional to the Fisher information matrix. It is also proved that the KLD is the only dissimilarity measure of two probability distributions, which satisfies the characterization of entropy [30].

The rest of this paper is organized as follows. Section 2 presents the formulation of ApEn and its proof showing that ApEn is equal to the entropy rate of a first-order stationary Markov chain. Section 3 describes how an image can be induced to a weighted graph using the KLD and the image histogram. Section 4 shows the computation of the entropy rate of a Markov chain of a KLD-weighted graph of an image, which can be used as a tool for quantifying the regularity or complexity of image data. Experiment results of image quality assessment obtained from the proposed method are presented in Section 5. Finally, Section 6 is the concluding remarks of the research findings.

2. ApEn and entropy rate of a Markov chain

The formulation of ApEn [23] is outlined as follows [19]. Let a time series or sequence $\mathbf{u} = (u_1, u_2, \dots, u_N)$. The whole sequence is first split into a set of vectors, each with a predetermined length m: $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N-m+1})$, where $\mathbf{x}_i = (u_i, u_{i+1}, \dots, u_{i+m-1})$, $i = 1, 2, \dots, N-m+1$. ApEn further requires a predetermined positive tolerance value r that is used for decision making if a pair of vectors are similar to each other. The probability of vector $\mathbf{x}_i$ that is similar to vector $\mathbf{x}_j$ is computed as

$$C_i(m,r) = \frac{1}{N-m+1} \sum_{j=1}^{N-m+1} \theta(d(\mathbf{x}_i, \mathbf{x}_j)), \tag{1}$$

where $\theta(d(\mathbf{x}_i, \mathbf{x}_j))$ is the Heaviside step function defined as

$$\theta(d(\mathbf{x}_i, \mathbf{x}_j)) = \begin{cases} 1 & : d(\mathbf{x}_i, \mathbf{x}_j) \le r \\ 0 & : d(\mathbf{x}_i, \mathbf{x}_j) > r \end{cases} \tag{2}$$

The distance between the two vectors can be obtained by

$$d(\mathbf{x}_i, \mathbf{x}_j) = \max_k (|u_{i+k-1} - u_{j+k-1}|), k = 1, 2, \dots, m. \tag{3}$$

The probabilities of all vectors being similar to one another are computed as

$$C(m,r) = \frac{1}{N-m+1} \sum_{i=1}^{N-m+1} \log(C_i(m,r)). \tag{4}$$

ApEn is expressed as a family of formulas, with fixed m and r, as

$$ApEn(m,r) = \lim_{N \to \infty} (C(m,r) - C(m+1,r)). \tag{5}$$

For the first-order stationary Markov chain of a discrete state space X, with $r < |x-y|$, $x \neq y$, where x and y are state-space values, $\forall m$, it is shown in [23] that ApEn is equal to the entropy rate of the Markov chain, which is

$$ApEn(m,r) = -\sum_{x \in X} \sum_{y \in X} \pi_x p_{xy} \log p_{xy}, \tag{6}$$

where π_x is the density function of x, and p_{xy} is the probability of transition from x to y.

The development of the entropy rate of a stationary Markov chain of a weighted graph of an image is described in the subsequent sections.