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Periodic solutions and their stability of some higher-order positively homogenous differential equations

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ABSTRACT

In the present paper we study periodic solutions and their stability of the m-order differential equations of the form

$x^{(m)} + f_n(x) = \mu h(t),$

where the integers $m, n \geq 2$, $f_n(x) = \delta x^n$ or $\delta |x|^n$ with $\delta = \pm 1$, $h(t)$ is a continuous T-periodic function of non-zero average, and μ is a positive small parameter. By using the averaging theory, we will give the existence of T-periodic solutions. Moreover, the instability and the linear stability of these periodic solutions will be obtained.

1. Introduction and statement of the main results

In this paper we are concerned with periodic solutions of some higher order homogenous or positively homogenous differential equations. A typical example is the second-order ordinary differential equation

$$\ddot{x} + x^3 = h(t). \tag{1.1}$$

In [10] Morris proved that if $h(t)$ is a T-periodic C^1 function and its average

$$\bar{h} := \frac{1}{T}\int_0^T h(t)dt$$

is zero, then Eq. (1.1) has periodic solutions of period kT for all positive integer k. Later on the same result was proved by Ding and Zanolin [5] without the assumption that $\bar{h} = 0$. More recently Ortega in [11] proved that Eq. (1.1) has finitely many stable periodic solutions of a fixed period.

Other authors have studied more general problems related with non-autonomous differential equations, as for instance: when a periodic solution or an equilibrium point of an autonomous differential system persists as a periodic solution if the autonomous differential system is periodically perturbed. Thus for dimension two and for an equilibrium Buică and Ortega in [3] characterized the persistence of such periodic solutions. The Brouwer degree theory was used by these authors for obtaining their results, for these kind of problems see also the paper of Capietto, Mawhin and Zanolin [4] and the references therein. Besides the existence of periodic solutions, Ortega and Zhang in [12] have also studied the stability of the periodic solutions. For higher-order differential equations, some related works are [7,13].

The objective of this paper is to extend the mentioned results on the periodic solutions of the second-order differential equation (1.1) to the m-order differential equations of the form

$$x^{(m)} + f_n(x) = \mu h(t), \tag{1.2}$$

where $x^{(m)}$ denotes the m derivative of $x = x(t)$ with respect to the independent variable t. Here the integers $m \geq 2$, $n \geq 2$,

$$f_n(x) = \delta x^n \quad \text{or} \quad f_n(x) = \delta |x|^n, \quad \delta = \pm 1, \tag{1.3}$$

$h(t)$ is a continuous T-periodic function with non-zero mean value

$$\bar{h} \neq 0, \tag{1.4}$$

and $\mu > 0$ is a positive small parameter.

Notice that the differential equations (1.2) are only continuous in t. So in order to study the periodic solutions of these kind of

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differential equations and their kind of stability, we cannot use the classical version of the averaging theory because it works for C^2 differential equations, we need the improvements done in [1,2].

The periodic solutions of these type of differential equations have been studied recently when $m=2$ in [8], and when $m=3$ in [9]. Here we extend the study of the periodic solutions of the differential equations (1.2) when $m\geq 2$ is an arbitrary integer, and we also analyze the kind of stability of the periodic solutions that we find. The techniques for studying the periodic solutions of the differential equations (1.2) and their kind of stability are mainly based on the averaging theory, and consequently are completely different from the ones used by Morris, Capietto, Mawhin and Zanolin in the above mentioned papers.

The stability of a periodic solution of the differential equations (1.2) is the stability of the fixed point of the Poincaré map associated to the periodic solution. Thus, if some eigenvalue of the fixed point has modulus larger than one, then the periodic solution is *asymptotically unstable*. If all the eigenvalues of the fixed point have moduli smaller than one, then the periodic orbit is *asymptotically stable*. For more information about this kind of stability and the differences with the Liapunov stability see for instance the book [6]. Notice that, when $m=2$, equations (1.2) are Lagrangian equations and the periodic solutions cannot be asymptotically stable. Hence we shall study the asymptotical instability and the linear stability of the periodic solutions obtained. Here a periodic solution $x=\varphi(t)$ of the equation

$$x^{(m)}+f(x)=h(t)$$

is *linearly stable* if all solutions $y(t)$ of the linearized equation

$$y^{(m)}+f'(\varphi(t))y=0$$

are bounded in the following sense

$$\sup_{t\in\mathbb{R}}\left(|y(t)|+|y'(t)|+\cdots+|y^{(m-1)}(t)|\right)<+\infty.$$

1.1. Statement of the main results

Our main results are the following two theorems.

Theorem 1.1. *Consider the m-order differential equations*

$$x^{(m)}+\delta x^n=\mu h(t), \tag{1.5}$$

where m,n,δ and $h(t)$ are as in (1.3) and (1.4). Then for $\mu>0$ sufficiently small the following statements hold.

(a) *If n is odd, then the differential equation (1.5) has one periodic solution $x(t,\mu)$ of period T such that*

$$x(0,\mu)=\mu^{1/n}(\delta\bar{h})^{1/n}+O\big(\mu^{(m+n-1)/(mn)}\big).$$

Moreover, when $m\geq 3$, or when $m=2$ and $\delta=-1$, the periodic solution is asymptotically unstable, and when $m=2$ and $\delta=+1$, the periodic solution is linearly stable.

(b) *If n is even and*

$$\delta=\operatorname{sign}(\bar{h}), \tag{1.6}$$

then the differential equation (1.5) has two periodic solutions $x_{\pm}(t,\mu)$ of period T such that

$$x_{\pm}(0,\mu)=\pm\mu^{1/n}(\delta\bar{h})^{1/n}+O\big(\mu^{(m+n-1)/(mn)}\big). \tag{1.7}$$

Moreover, when $m\geq 3$, these periodic solutions are asymptotically unstable, and when $m=2$, one periodic solution $x_{-\delta}(t,\mu)$ is asymptotically unstable and the other one $x_{\delta}(t,\mu)$ is linearly stable.

Theorem 1.2. *Consider the m-order differential equations*

$$x^{(m)}+\delta|x|^n=\mu h(t), \tag{1.8}$$

where m,n,δ and $h(t)$ are as before. If (1.6) is satisfied, then for $\mu>0$ sufficiently small, the differential equation (1.8) has two periodic solutions $x_{\pm}(t,\mu)$ of period T which also satisfy (1.7). Moreover, when $m\geq 3$, these periodic solutions are asymptotically unstable, and when $m=2$, one periodic solution $x_{-\delta}(t,\mu)$ is asymptotically unstable and the other one $x_{\delta}(t,\mu)$ is linearly stable.

These theorems show that the linearly stable periodic solutions resulted from average theory can be obtained only in case $m=2$. In this case, the linearization equation is elliptic, i.e., the Floquet multipliers are different from ± 1 and have modulus 1.

Theorems 1.1 and 1.2 include as particular cases the results of [8,9]. Their proofs are given in Sections 2.2–2.5.

2. Proofs

We shall study the periodic solutions of the m-order periodically-driven ordinary differential equations (1.2). We remark that $f_n(x)=\delta x^n$ and $f_n(x)=\delta|x|^n$ are positively homogeneous

$$f_n(cx)\equiv c^n f_n(x)\qquad \forall c\geq 0. \tag{2.1}$$

2.1. Equations in the normal form for applying the averaging theory

By setting

$$x_i=x^{(i-1)},\qquad i=1,2,\cdots,m,$$

where for $i=1$ we define $x_1=x^{(0)}=x$, Eq. (1.2) is equivalent to the system

$$\begin{aligned} x_i'&=x_{i+1},\qquad i=1,2,\cdots,m-1,\\ x_m'&=\mu h(t)-f_n(x_1). \end{aligned} \tag{2.2}$$

We rescale system (2.2) using

$$x_i=\varepsilon^{i-1+m/(n-1)}X_i,\qquad i=1,2,\cdots,m,\qquad \mu=\varepsilon^{m+m/(n-1)}, \tag{2.3}$$

where ε is a positive parameter. Then (2.2) is reduced to the following system

$$X'=\varepsilon F_n(t,X), \tag{2.4}$$

where

$$\begin{aligned} X&=(X_1,X_2,\cdots,X_m)\in\mathbb{R}^m,\\ F_n(t,X)&=(X_2,\cdots,X_m,h(t)-f_n(X_1))\in\mathbb{R}^m. \end{aligned}$$

Here the positive homogeneity (2.1) for $f_n(x)$ is used.

System (2.4) is written in the standard normal form for applying the averaging theory in order to study its periodic solutions and their stability, see [1,2] for more details on the averaging theory here used.

2.2. Zeros of the average function

In order to apply the averaging theory for studying the periodic solutions of the differential system (2.4) and consequently of the differential equation (1.2) we must study the zeros of the average function associated to system (2.4).

For any $X\in\mathbb{R}^m$ we have

$$\bar{F}_n(X):=\frac{1}{T}\int_0^T F_n(t,X)dt\equiv\begin{pmatrix} X_2\\ \vdots\\ X_m\\ \bar{h}-f_n(X_1)\end{pmatrix}.$$

One sees that $\bar{F}_n(X)$ is C^1 in $X\in\mathbb{R}^m$. Moreover X_* is a zero of $\bar{F}_n$ if and only if

$$X_*=(x_*,0,\cdots,0),$$