Chaos, Solitons and Fractals 106 (2018) 289–294

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Soliton solutions and eigenfunctions of linearized operator for a higher-order nonlinear Schrödinger equation

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ARTICLE INFO

Article history:
Received 3 October 2017
Accepted 14 November 2017
Available online 6 December 2017

MSC:
35Q55
37K10

Keywords:
Partial differential equations
Higher-order Schrödinger equation
Linearization operator
Squared eigenfunctions
Soliton perturbation theory

ABSTRACT

The Inverse Scattering Transform (IST) method is applied to find soliton solutions for a higher-order nonlinear Schrödinger (NLS) equation. Eigenfunctions of linearized operator which have a central role in soliton perturbation theory are explicitly found.

1. Introduction

Partial Differential Equations (PDEs) have a central role in pure and applied sciences where a few ones have analytical solutions. Amongst those have analytical solutions soliton equations have vast application in physics, telecommunication and differential geometry. Technically, soliton equation is a PDE with soliton solution. A soliton has the property that tends to zero where the spatial variable approaches infinity. Recently, it has been shown that a soliton equation can be recognized due to be as the compatibility condition of a so-called *Lax pair,i.e.*, see Section 2. Utilizing the Lax pair and applying the soliton property, the general soliton solution of an equation can be constructed. This method is so-called Inverse Scattering Transform (IST) which has been developed and applied for several soliton (integrable) equations. Subsequently, Ablowitz et al. [1] introduced a large class of integrable equations. For a detailed review of IST for nonlinear Schrödinger (NLS) equation and AKNS hierarchy, we refer the readers to Yang [8].

Adjusting the scattering operator associated with the hierarchy and determining the member index number in the AKNS procedure lets higher-order and vector integrable equations be easily constructed. For a practical example Doktorov et al. [6] considered an integrable matrix version related to NLS equation and showed that it is equal to the integrable bright spinor Bose–Einstein condensates model with integer spin $F=1$ (after some re-scalings). The matrix NLS equation can be re-constructed via the AKNS where the related Lax pair is readily available and therefore IST can be developed to find soliton solutions. Ahmadi and Hoseini [2] also developed the soliton perturbation theory for the matrix NLS equation where the explicit forms of eigenstates of the related linearized operator are needed. They proved that the closure set corresponding to the matrix NLS equation contains 8 localized (continuous) and 6 non-localized (discrete) eigenstates.

The authors in [4] constructed a higher-order NLS (HNLS) equation

$$iu_t = u_{xxxx} + 4|u_x|^2 u + 8|u|^2 u_{xx} + 6u^* u_x^2 + 2u^2 u_{xx}^* + 6|u|^4 u, \qquad (1)$$

and examined weak interaction for two well-separate single solitons. Note that here "$*$" represents complex conjugation. Similarity between (1) and scalar NLS equation lets the one-soliton solution of (1) be easily guessed. They determined the linearized operator around rank-one soliton solution and find explicitly all continuous and discrete eigenfunctions of the operator. Despite of complicated form of the operator it was shown that the eigenstates and also adjoint eigenstates are exactly the same as those related to NLS equation.

One of the recent development in soliton theory is soliton perturbation theory where the solitary wave solution of (small) correction to the integrable equation can analytically be determined up to any order of perturbation importance factor. This requires

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https://doi.org/10.1016/j.chaos.2017.11.021
0960-0779/

that the complete set of eigenfunctions for the linearized problem, related to the nonlinear wave equation, be determined. Yang [7] constructed this set for a large class of integrable nonlinear wave equations such as the Kortewegde Vries (KdV), NLS and modified KdV (mKdV) equations. The same procedure can be exploited to find the eigenstates of the adjoint linearization operator. He found that the eigenfunctions for these hierarchies are the squared Jost solutions. Chen and Yang [3] developed direct soliton perturbation theory for the derivative NLS and the modified NLS equations. Using the similarity between the KdV and derivative NLS hierarchies they showed that the eigenfunctions for the linearized derivative NLS equation are the derivatives of the squared Jost solutions. This is in contrast to the counterpart for NLS, Hirota and mKdV hierarchies, where the eigenfunctions are just the squared Jost solutions. Suppressing the secular terms, they also found the slow evolution of soliton parameters and the perturbation-induced radiation.

In the present paper, the IST is developed for HNLS (1) and it will be shown that an analogous manner for NLS equation can be exploited to find the soliton solutions. The reason for similarity is that the spatial equations of Lax pairs for NLS and HNLS (1) are exactly the same. The only different part of the procedure is the time evolution for soliton which is caused by different temporal equations of Lax pairs. We adopt the notations in [8] through the paper to show that HNLS (1) can be treated as NLS equation despite its more complicated form. Using the spatial equation of Lax pair, we also find the squared eigenfunctions of the Zakharov–Shabat system and prove the closure relation for the eigen-states related to (1) and finally we find explicit forms of eigenfunctions for linearized operator and the adjoint.

This paper contains 3 sections. In Section 2, the IST procedure is reviewed for HNLS (1) using the explicit forms of Lax pair. We find the most general soliton solution and determine explicitly rank-one soliton solution as an example. In Section 3, the squared eigenfunctions for the Zakharov-Shabat system related to HNLS (1) are found based on Jost solutions. These squared eigenfunctions contain those related to linearized operator and its adjoint. In Section 4, the explicit forms of the eigenfunctions of the linearized operator around one-soliton solution are constructed. And finally, Section 5 concludes the results of the paper.

2. IST for (1)

It is well known that to establish a general soliton solution of a soliton equation via IST, a pair of ordinary differential equations (ODEs) named Lax pair is needed where their compatibility condition is the soliton equation. Thanks to pioneer work of Ablowitz et al.[1], not only a large family of integrable equations has been found, but also their corresponding Lax pairs can easily be constructed. We begin with Lax pair for HNLS (1) and mention its relationship with HNLS (1). Note that HNLS (1) is the next even member of the NLS integrable hierarchy.

If we define two ODEs (Lax pair)

$$Y_x = MY, \quad Y_t = NY, \tag{2}$$

where

$$M = -i\zeta\Lambda + Q, \quad N = -8i\zeta^4\Lambda + 8\zeta^3 Q + 4i\zeta^2 V + R,$$

and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}$, $V = \begin{pmatrix} |u|^2 & u_x \\ u_x^* & -|u|^2 \end{pmatrix}$,

$$R = \begin{pmatrix} |u_x|^2 - uu_{xx}^* - u_{xx}u^* - 3|u|^2 & -(u_{xxx} + 6|u|^2 u_x) \\ -(u_{xxx}^* + 6|u|^2 u_x^*) & -(|u_x|^2 - uu_{xx}^* - u_{xx}u^* - 3|u|^2) \end{pmatrix},$$

and ζ is scattering parameter, then the compatibility condition for (2),i.e., $Y_{xt} = Y_{tx}$ gives

$$M_t - N_x + [M, N] = 0, \tag{3}$$

which is HNLS (1). Here

$$[M, N] = MN - NM, \tag{4}$$

is the commutator bracket.

A soliton has a key property that decays to zero sufficiently fast as the spatial variable x approaches infinity. We shall use this property to find the fundamental solution of (2). Therefore, is clear that $Y \propto \exp\{-i\zeta\Lambda x - 8i\zeta^4\Lambda t\}$ as $x \longrightarrow \pm\infty$. It is convenient that (2) be changed to

$$J_x = -i\zeta[\Lambda, J] + QJ, \tag{5}$$

$$J_t = -8i\zeta^4[\Lambda, J] + (8\zeta^3 Q + 4i\zeta V + R)J, \tag{6}$$

via

$$Y = Je^{-i\zeta\Lambda x - 8i\zeta^4\Lambda t}, \tag{7}$$

where the Jost solution J is (x, t)-independent at infinity.

As the main step in IST we mainly focus on the first equation of (37) called Zakharov–Shabat system and time evolution of the solitons shall be done by applying the second ODE in (37) when they explicitly determined. As mentioned earlier, HNLS (2) shares its Zakharov–Shabat system with NLS and hence the spatial evolution using IST procedure for equations will be analogous. We consider two Jost solutions $J_\pm(x, \zeta)$ of the scattering problem (5), with the following asymptotic

$$J_\pm(x, \zeta) \longrightarrow I, \qquad x \longrightarrow \pm\infty, \tag{8}$$

where I is the 2×2 unit matrix. Note that we temporally forget “t′′ from the notations. Abel’s identity shows that

$$\det J_\pm(x, \zeta) = 1, \tag{9}$$

for all (x, ζ). As $J_\pm E$ ($E = e^{-i\zeta\Lambda x}$) are both solutions of the (linear) Zakharov–Shabat system, they are linearly related as $J_- E = J_+ ES$, where $S = S(\zeta)$ is called scattering matrix and the potential u can be retrieved from the elements of S. Consequently, the property (9) gives

$$\det S(\zeta) = 1, \qquad \zeta \in \mathbb{R}. \tag{10}$$

Up to now, the scattering parameter ζ was considered on real line. Extending the analyticity of Jost solutions $J_\pm(x, \zeta)$ and therefore $S(\zeta)$ to the ζ-complex half-planes provides the opportunity of applying Riemann–Hilbert Problem to construct the soliton solutions for HNLS (1) from the scattering data stored in S. For example, it can be shown that the first column of J_- and the second column of J_+ can be analytically continued to the upper half plane $\zeta \in \mathbb{C}_+$, while the second column of J_- and the first column of J_+ can be analytically continued to the lower half plane $\mathbb{C}_-$, for a detailed review of application of Volterra integral equations see [8] and [9]. For simplicity, we let $\Phi = J_- E$ and $\Psi = J_+ E$ and express (Φ, Ψ) as a collection of columns and Φ^{-1}, Ψ^{-1} as a collection of rows as

$$\Phi = (\phi_1, \phi_2), \quad \Psi = (\psi_1, \psi_2), \quad \Phi^{-1} = \begin{pmatrix} \widehat{\phi_1} \\ \widehat{\phi_2} \end{pmatrix}, \quad \Psi^{-1} = \begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix}. \tag{11}$$

Finally, collecting the functions with same analyticity areas yields that the Jost solutions

$$P^+ = (\phi_1, \psi_2)e^{i\zeta\Lambda x} = J_- H_1 + J_+ H_2, \quad \textit{where} \quad H_1 \equiv diag(1, 0), \quad H_2 \equiv diag(0, 1), \tag{12}$$

are analytic in $\zeta \in \mathbb{C}_+$, and in a similar consideration the Jost solutions

$$P^- = e^{-i\zeta\Lambda x}\begin{pmatrix} \widehat{\phi_1} \\ \widehat{\psi_2} \end{pmatrix} = H_1 J_-^{-1} + H_2 J_+^{-1} \tag{13}$$