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On the effective interfacial resistance through quasi-filling fractal layers

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ABSTRACT

This paper concerns the periodic homogenization of the stationary heat equation in a domain with two connected components, separated by an oscillating interface defined on prefractal Koch type curves. The problem depends both on the parameter ε that defines the periodic structure of the interface and on n, which is the index of the prefractal iteration. First, we study the limit as ε vanishes, showing that the homogenized problem is strictly dependent on the amplitude of the oscillations and the parameter appearing in the transmission condition. Finally, we perform the asymptotic behaviour as n goes to infinity, giving rise to a limit problem defined on a domain with fractal interface.

1. Introduction

In this paper, we are interested in the periodic homogenization of the stationary heat equation in a two-component domain with oscillating interface. The macroscopic modelling of heat transfer in composites with interfacial thermal resistance has been derived in [2] and then considered in [14,17,21,30]. For domains with an oscillating interface having a fixed amplitude of oscillations we mention [5,19,29]; for boundary value problems with rapidly alternating boundary conditions, see the book [12] and the references therein.

Actually, in this work we consider the boundary value problem associated to an elliptic equation in the set $Q=(0,1)\times(-\frac{1}{2},\frac{1}{2})\subset\mathbb{R}^2$, separated in two connected components, each one containing an heterogeneous material. Such a problem has been already analyzed in [16], where the internal layer dividing the domain is represented as the graph of a rapidly oscillating function of period ε, defined on an hyperplane.

The novelty of this article w.r.t. [16]. is that here we consider the rapidly oscillating internal layer defined along the prefractal Koch curves. We recall that the prefractal curves tend to the Koch type fractals in the Hausdorff metric, when $n\to+\infty$. Moreover, while the prefractal curves have dimension equal to one, the limit fractals have a Hausdorff dimension between 1 and 2: for this reason, we call them quasi-filling curves (see Section 2 and Fig. 1).

For every given non-negative integer n and every small ε, we define the interface $\Gamma_{\varepsilon,n}$ associated to the prefractal set K_n by means of an iterative procedure applied to

$$\Gamma_\varepsilon=\left\{x\in Q: x_2=\varepsilon^k g\left(\frac{x_1}{\varepsilon}\right)\right\},$$

being $g:[0,1]\to\mathbb{R}$ a periodic positive Lipschitz continuous function (see Section 3). The amplitude of the oscillations of Γ_ε is of order ε^k, with $k\geq 1$. Differently from [16], the case $0<k<1$ cannot be considered due to the prefractal nature of the interface $\Gamma_{\varepsilon,n}$ (see Section 3 again). We call $Q^+_{\varepsilon,n}$ and $Q^-_{\varepsilon,n}$ respectively the upper and lower parts of Q separated by $\Gamma_{\varepsilon,n}$ (see Fig. 2).

On the interface, we prescribe the continuity of the conormal derivatives and a jump of the solution, which is proportional to the conormal derivative throughout a function of order ε^γ, $\gamma\in\mathbb{R}$. The boundary value problem can be formally written as

$$\begin{cases}-\mathrm{div}(A^\varepsilon\nabla u_{\varepsilon,n})=f \quad \text{in} \quad Q\setminus\Gamma_{\varepsilon,n}\\ (A^\varepsilon\nabla u_{\varepsilon,n})^+\cdot\nu_{\varepsilon,n}=(A^\varepsilon\nabla u_{\varepsilon,n})^-\cdot\nu_{\varepsilon,n} \quad \text{on} \quad \Gamma_{\varepsilon,n}\\ (A^\varepsilon\nabla u_{\varepsilon,n})^+\cdot\nu_{\varepsilon,n}=-\varepsilon^\gamma\sigma_n(u^+_{\varepsilon,n}-u^-_{\varepsilon,n}) \quad \text{on} \quad \Gamma_{\varepsilon,n}\\ u_{\varepsilon,n}=0 \quad \text{on} \quad \partial Q\end{cases}, \qquad (1)$$

where $\nu_{\varepsilon,n}$ is the unit outward normal to $Q^+_{\varepsilon,n}$ on $\Gamma_{\varepsilon,n}$, $f\in L^2(Q)$, $A^\varepsilon=A\left(\frac{x}{\varepsilon}\right)$ is a periodic bounded elliptic matrix field and σ_n is a constant whose role is clarified in Section 5. We denote with the superscripts + and - the restrictions respectively to the upper and lower part of Q w.r.t. the interface.

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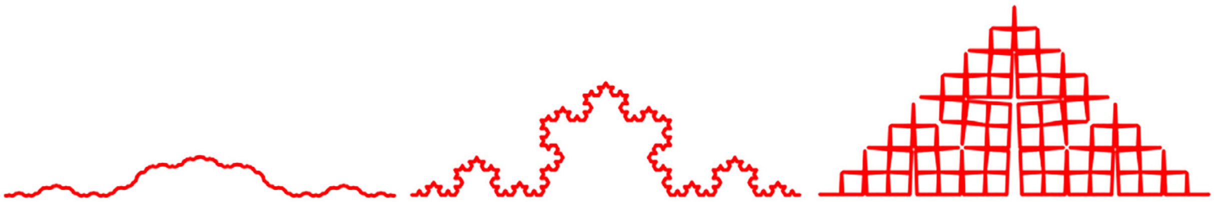

Fig. 1. Prefractal Koch curves with $n = 4$ and $\ell = 3.8$, $\ell = 3$, $\ell = 2.2$ respectively.

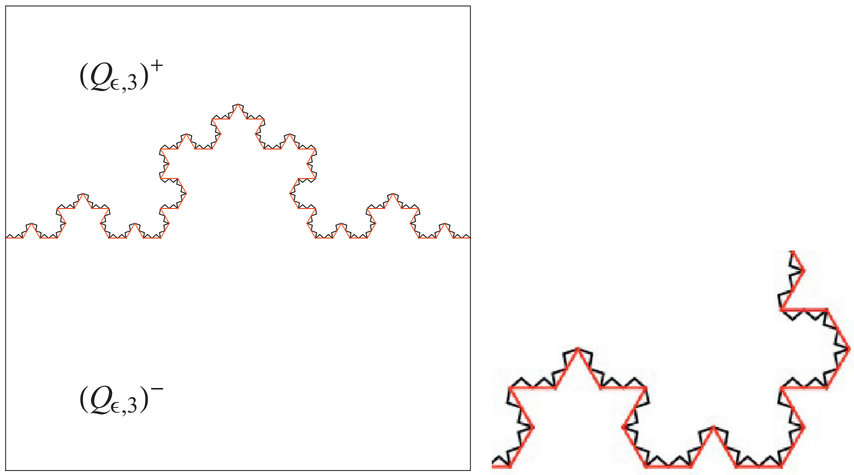

Fig. 2. The domain Q with the interface $\Gamma_{\varepsilon,3}$ (left), a detail of $\Gamma_{\varepsilon,3}$ (right).

The *physical phenomenon* that this mathematical problem models is the stationary heat diffusion in a region of two composite materials having an imperfect contact between them. The corresponding parabolic problem describes heat exchange between two media prepared initially at different temperatures and separated by a partially isolating boundary. The transmission boundary conditions on the interface impose the continuity of the temperature flux across the boundary, and relate this flux to the temperature drop at the boundary due to thermal isolation (see page 23 of [10]). The quantity $\varepsilon^{\gamma}\sigma_n$, which appears in the jump condition of the temperature on the interface, is usually referred as the interfacial thermal conductance or the coefficient of interface heat transfer (see page 19 in [10]). More precisely, σ_n depends on the structural constants of the self-similar fractal, i.e., the number of the contractive similarities and the value of contraction factor (this choice of σ_n is crucial only when we perform the asymptotic analysis for $n\to\infty$). The quantity ε^{γ} clarifies how the interfacial thermal conductance depends on the parameter ε that defines the periodic structure that gives rise to an imperfect contact between the two components.

First we focus our study on the homogenization of problem (1) as ε vanishes, freezing the fractal iteration at a fixed, but arbitrary, stage n (see [25] for non-periodic homogenization for conductive layers of prefractal type) and then, we perform the asymptotic analysis as $n\to\infty$.

We point out that the main difficulty in treating the problem (1) is that $\Gamma_{\varepsilon,n}$ cannot be seen as the graph of a function. To our knowledge, this is the first work on periodic homogenization with fractal structures (homogenization with small perforations of complicated shapes has been studied in [15]). The motivation for our work lies in the fact that many phenomena in physics, physiology, electrochemistry and chemical engineering are modeled by irregular layers; the peculiar aspect is that these phenomena take place in domains with small bulk and large interfaces (see [20] for a detailed discussion).

For example, as investigated in [3], the study of this type of problem is relevant to improve heat exchangers, e.g. cooling of metallic radiators or thermal isolation of pipes and buildings. In fact, depending on application, cooling rate has to be either enhanced (e.g. in the case of microprocessors or nuclear reactors), or slowed down (e.g. in the case of pipes and buildings). For these purposes, it is therefore crucial to understand how the shape of the boundary influences heat exchange and how an irregular (e.g. fractal) boundary with a very large exchange area significantly speeds up cooling.

The plan of the paper is the following.

In Section 2, we give some preliminaries about Koch curve type fractals, in order to setup the problem (1) in Section 3.

Section 4 is devoted to homogenization of problem (1) in the limit as ε vanishes, freezing the fractal iteration at a fixed, but arbitrary, stage n. We prove in Theorems 4.1 and 4.2 that, as ε goes to zero, the value of γ plays a fundamental role in the asymptotic behaviour of the solution of problem (1). Indeed, if $\gamma < 0$, the homogenized problem (17) is the same as the one without interface. If $\gamma > 0$, it is equivalent to two independent Neumann problems, thus K_n is an isolating interface (homogenized problem (16)). Finally, in the case $\gamma = 0$, we have two different situations based on the value of k. If $k = 1$, the homogenized problem (14) consists of a homogenized diffusion equation and a transmission condition with a jump depending on the shape of the function g; on the other hand, in the case $k > 1$ the shape of g does not contribute (problem (19)).

We point out that the behavior of homogenized problems with respect the values of γ and k is the same as in the paper [16].

In Section 5 we perform the asymptotic analysis of the previous homogenized problems as $n\to+\infty$. More precisely, Theorems 5.1 and 5.2 assert that solutions of the homogenized problems converge respectively to the solutions of the corresponding problems with Koch fractal interfaces as $n\to+\infty$. The value of σ_n is crucial for these limits.