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Analysis and application of new fractional Adams–Bashforth scheme with Caputo–Fabrizio derivative

Kolade M. Owolabi*, Abdon Atangana

Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa

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ABSTRACT

Recently a new fractional differentiation was introduced to get rid of the singularity in the Riemann-Liouville and Caputo fractional derivative. The new fractional derivative has then generate a new class of ordinary differential equations. These class of fractional ordinary differential equations cannot be solved using conventional Adams–Bashforth numerical scheme, thus, in this paper a new three-step fractional Adams–Bashforth scheme with the Caputo–Fabrizio derivative is formulated for the solution linear and nonlinear fractional differential equations. Stability analysis result shows that the proposed scheme is conditionally stable. Applicability and suitability of the scheme is justified when applied to solve some novel chaotic systems with fractional order $\alpha \in (0, 1)$.

1. Introduction

The primary interest of this paper is to extend the analysis and application of the newly developed fractional two-step Adams–Bashforth scheme with the Caputo–Fabrizio derivative to nonlinear system of the form

$$\begin{aligned} &\mathcal{D}^{\alpha}_{0,t}u_i(t) = f_i(u_i, t),\\ &\quad u_i(0) = \vartheta_i, \quad i = 1, 2, \ldots, n \end{aligned} \tag{1.1}$$

where $\mathcal{D}^{\alpha}_{0,t}$ denotes the α_i order derivative of function $u_i(t)$, for $0 < \alpha_i \leq 1$. This derivative is replaced with the Caputo–Fabrizio fractional derivative defined by

$$\begin{aligned} {}^{CF}\mathcal{D}^{\alpha}_{0,t}(u_i(t)) = \frac{M(\alpha)}{1-\alpha}\int_{0^t} u_i'(\xi)\exp\left[-\alpha\frac{t-\xi}{1-\alpha}\right]d\xi,\\ i = 1, 2, \ldots, n. \end{aligned} \tag{1.2}$$

Most fractional differential equations describing real-world (physical) problems are highly complicated and cannot sometimes be solved analytically. A lot of numerical approaches in connection with derivatives of fractional order describing these real-world problems alter essentially in the many in which the derivative of fractional order is tailored, see for instance [3–6,13–15,18,32,34,35] and references therein. Numerical approximation of a derivative of fractional order has a highly complicated formula compared to those of integer order due to their nonlocal nature, and therefore the calculation at a particular point requires knowledge of the function further out of the region close to that point. Accordingly, finite difference approximations of derivatives of fractional order engage a quantity of points that alters according to how faraway we are from the border line.

One of the most recent fractional order derivatives was proposed by Caputo and Fabrizio [10,11], where it was shown that the new-fangled derivative contains additional encouraging properties in comparison with the older version. For example, they have shown that it can represent substance heterogeneities and configurations with different scales, which clearly cannot be overseeing with the prominent local theories and also the known fractional derivative. Another application is in the investigation of the macroscopic behaviors of some materials that are associated with nonlocal communications between atoms, which are recognized to be important of the properties of material [3].

* Corresponding author.
E-mail addresses: kmowolabi@futa.edu.ng (K.M. Owolabi), atanganaa@ufs.ac.za (A. Atangana).

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Ordinary differential equations (ODEs), partial differential equations (PDEs) and partial integro-differential equations (PIDEs) of fractional order have been the major subject of activities in the last few decades, as a result of their frequent appearance in various applications in biology, economics and finance, engineering, fluid mechanics, physics and viscoelasticity [18,34]. Consequently, a lot of attention has been devoted to the solutions of fractional ODEs, PIDEs of physical interest [9,12,25–30]. As widely observed, most of these nonlinear fractional differential equations do not either admit a closed form solution, or that the analytical solution is too involved to be useful, which perhaps ruled out a closed form solution. In such a situation, approximation and numerical method must be proposed. Among several other methods that have been used include the finite difference method [9,16,21,26], spectral methods [9,27,28,31,33,40,41] the variation iteration method [22–24], and the Adomian decomposition method [22,36,39].

With the wider applicability of the new class of fractional differential equation from Caputo–Fabrizio derivative with exponential decay law kernel, new, accurate and efficient numerical schemes are needed. A very powerful numerical scheme for solving nonlinear ordinary differential equation known as Adams–Bashforth method has been used in several instances to handle many chaotic models. This model is being recognized as a stable method for nonlinear ordinary differential equations and also chaotic models arising in biology and other fields of science. However, this model has not being developed for nonlinear models with Caputo–Fabrizio derivative. This work is therefore devoted to the development of the Adams–Bashforth scheme for ordinary differential equation with local fading memory induced by the Caputo–Fabrizio fractional differential operators.

The rest of this paper is organized as follows. Some basic definitions and properties of fractional calculus are presented in brief in Section 2. The new three-step Adams–Bashfort method with the Caputo fractional derivative, and the stability condition are derived in Section 3. Some numerical experiments are given in Section 4 to justify the applicability of the proposed method. We conclude the paper with Section 5.

2. Basic definitions and properties of fractional calculus

In this section, we report a quick tour of some of the useful basic definitions and properties of the fractional calculus ranging from local to nonlocal case [4,15,37,38].

Definition 2.1. A real function $u(t)$, $t>0$ is said to be in space C_ν, $\nu\in\mathbf{R}$ if there exists a real number $k(>\nu)$, such that $u(t)=t^k u_1$, where $u_1(t)\in C[0,\infty)$, which is alsoin space C_ν^n if and only if $u^{(n)}\in C_\nu$, $n\in N$.

Till date, the most popularly used fractional derivatives remain the Riemann–Liouville and the Caputo types defined as

$$ {}^{RL}\mathcal{D}^{\alpha}_{0,t}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\xi)^{n-\alpha-1}u(\xi)d\xi \tag{2.3}$$

and

$$ {}^{C}\mathcal{D}^{\alpha}_{0,t}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}\frac{d^n}{dt^n}u(\xi)d\xi \tag{2.4}$$

respectively, with $n-1<\alpha\le n$. Most new fractional derivatives that are recently developed hang on the shoulder of these two derivatives. For instance, the Atangana–Baleanu fractional derivative in the sense of Caputo and Riemann–Liouville derivatives.

Definition 2.2. Let $u\in H^1(a,\ b), a<b,\ \alpha\in[0,\ 1]$ then, the definition of the Atangana and Baleanu fractional derivative in Caputo sense is given as [2,7]

$$ {}^{ABC}\mathcal{D}^{\alpha}_{a,t}(u(t))=\frac{M(\alpha)}{1-\alpha}\int_a^t u'(\xi)E_\alpha\left[-\alpha\frac{(t-\xi)^\alpha}{1-\alpha}\right]d\xi \tag{2.5}$$

where a is the starting point, usually assumed to be zero, and $M(\alpha)$ has the same properties as in the case of the Caputo–Fabrizio [10,11] fractional derivative.

Definition 2.3. Let $u\in H^1(a,\ b), a<b,\ \alpha\in[0,\ 1]$ then, the definition of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense becomes [2,7]

$$ {}^{ABR}\mathcal{D}^{\alpha}_{0,t}(u(t))=\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t u(\xi)E_\alpha\left[-\alpha\frac{(t-\xi)^\alpha}{1-\alpha}\right]d\xi \tag{2.6}$$

Obviously, both Eqs. (2.5) and (2.6) have a non-local kernel. Also, one obtains zero whenever the function in Eq. (2.5) is constant. In both cases, the term E_α is referred to as the one-parameter Mittag–Leffler function, given by

$$ E_\alpha(u)=\sum_{k=0}^{\infty}\frac{u^k}{\Gamma(\alpha k+1)},\quad \alpha>0,\quad \alpha\in\mathbb{R},\quad u\in\mathbb{C}. \tag{2.7}$$

Definition 2.4. Let u be a function in $H^1(a,\ b)$; $b>a; \alpha\in[0,\ 1]$ then, the Caputo–Fabrizio fractional derivative of order α is defined as [10]

$$ {}^{CF}\mathcal{D}^{\alpha}_{0,t}u(t)=\frac{M(\alpha)}{(1-\alpha)}\int_0^t u'(\xi)\exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right]d\xi, \tag{2.8}$$

where $M(\alpha)$ denotes a normalized function, such that $M(0)=M(1)=1$.

But, in case the function u does not belong to $H^1(a,\ b)$, the Caputo–Fabrizio derivative for this version is defined as

$$ {}^{CF}\mathcal{D}^{\alpha}_{0,t}u(t)=\frac{\alpha M(\alpha)}{(1-\alpha)}\int_0^t (u(t)-u(\xi))\exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right]d\xi. \tag{2.9}$$

The above Caputo–Fabrizio fractional derivative was later modified by Losada and Nieto in [20] as

$$ {}^{CF}\mathcal{D}^{\alpha}_{0,t}u(t)=\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t u'(\xi)\exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right]d\xi. \tag{2.10}$$

3. Numerical method of approximation

In this section, we first introduce the two-step Adams–Bashforth scheme with the Caputo–Fabrizio fractional derivative, as proposed by Atangana and Owolabi [7].

We start by presenting the numerical approximation based on the definition of the Caputo–Fabrizio derivative for Caputo-type [1,10,11],

$$ {}^{CF}_{0}\mathcal{D}^{\alpha}_{t}(u(t))=\frac{M(\alpha)}{1-\alpha}\int_0^t u'(\xi)\exp\left[-\alpha\frac{t-\xi}{1-\alpha}\right]d\xi. \tag{3.11}$$

For some integer $N>0$, the grid size in time for finite difference technique is given by

$$k=\frac{1}{N}.$$

In the time interval $[0,\ T]$, the grid points are denoted as $t_n=nk,\ n=0,1,2,\ldots,TN$. The value of the function u at the grid point is $u_i=u(t_i)$.

A discrete approximation to the Caputo–Fabrizio derivative of fractional order is given in [3] by the simple quadrature formula as

$$ {}^{CF}_{0}\mathcal{D}^{\alpha}_{t}(u(t_n))=\frac{M(\alpha)}{1-\alpha}\int_0^{t_n} u'(\xi)\exp\left[-\alpha\frac{t_n-\xi}{1-\alpha}\right]d\xi. \tag{3.12}$$